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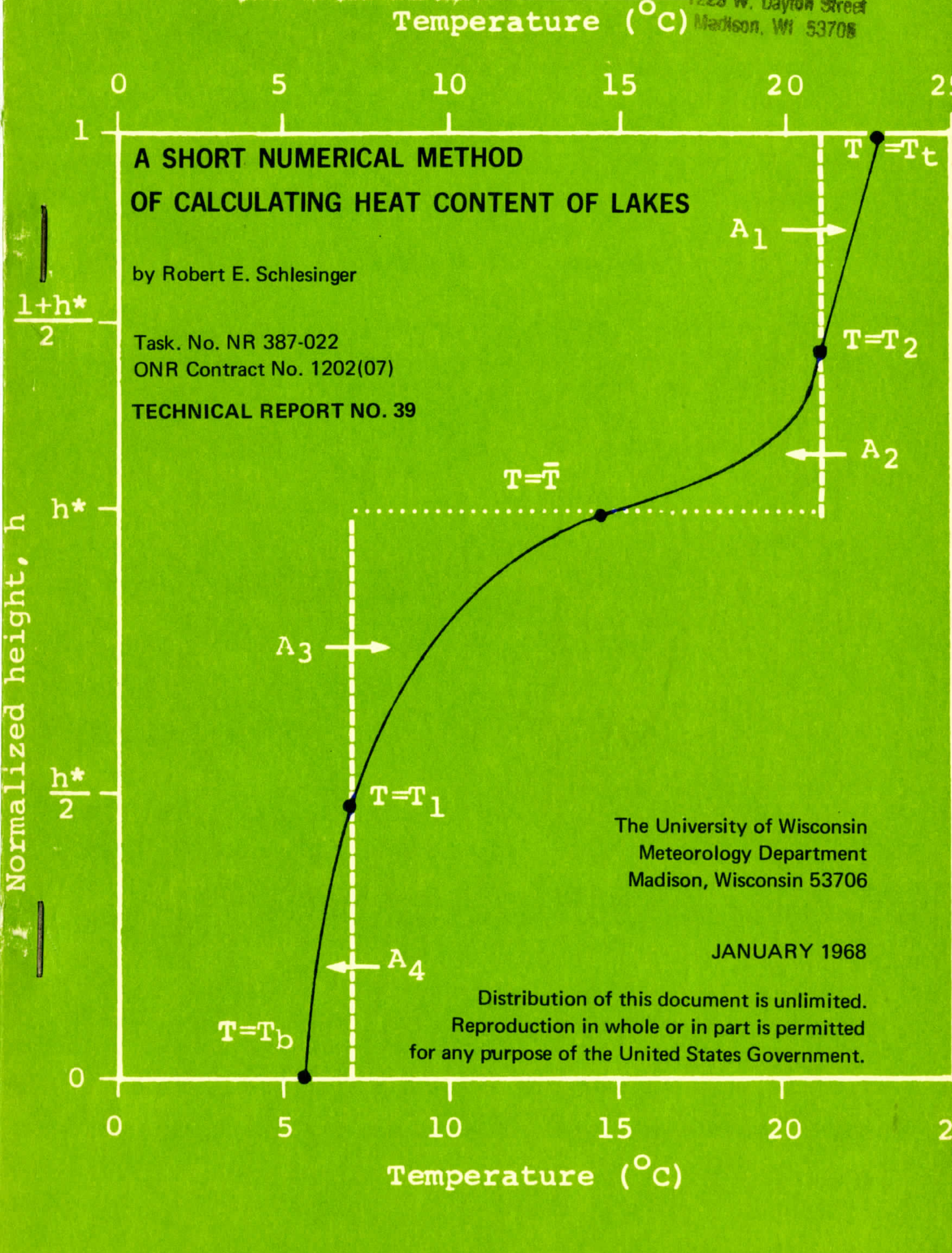
A SHORT NUMERICAL METHOD OF CALCULATING HEAT CONTENT OF LAKES

by Robert E. Schlesinger

Task. No. NR 387-022
ONR Contract No. 1202(07)

TECHNICAL REPORT NO. 39



The University of Wisconsin
Meteorology Department
Madison, Wisconsin 53706

JANUARY 1968

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A SHORT NUMERICAL METHOD

OF CALCULATING HEAT CONTENT OF LAKES

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I. Introduction

In recent years, studies of temperate lakes have provided strong quantitative evidence that a lake acts as an integrator of the climate of its surroundings. This is intuitively clear if one realizes that the gains and losses of heat and the forces which distribute it in the lake are largely controlled by such climatic factors as solar radiation, air temperature and wind. Dutton and Bryson (1960) have found a linear correlation of .98 between the surface temperature of Lake Mendota and the two-week running mean air temperature during one year, and McFadden (1965) has shown that the thawing and freezing dates of Canadian lakes correspond closely to the dates of certain critical values of forty-day running mean air temperatures.

This paper will focus on the heat content of a lake, defined by Bryson and Dutton (1960) to be the number of calories per square centimeter at the surface required to warm the lake from a uniform temperature of $0^{\circ}C$ to a given temperature distribution. In mathematical terms,

$$\theta = \int_0^H c\,T(z)\,w(z)\,dz \qquad (1)$$

where θ denotes heat content, z is the vertical distance from the deepest point of a lake of total depth H cm, c is the heat capacity per unit volume of pure water in calories per ^{o}C per cm^{3}, T(z) is the mean temperature in ^{o}C over the cross-sectional area at a given level and w(z) is the ratio of cross-sectional area at that level to the surface area. Numerically, c can be taken to be unity with ample accuracy for heat content calculations over the range of temperatures observed in lakes.

Although a uniform temperature of 0^{o}C is practically never observed in a lake, it is permissible to use 0^{o}C as a reference temperature in equation (1) since changes of θ with time rather than values of θ itself are of importance to the heat budget of a lake, and differences in θ are unchanged if T is replaced by $T-T_o$ for any fixed temperature T_o higher than 0^{o}C. The value $T_o = 4^{o}$C, the temperature of maximum density for pure water, is of special physical significance to the mechanism of warming a temperate lake. At temperatures below 4^{o}C, heat delivered to the uppermost layers by solar radiation is distributed throughout the lake by the sinking and circulation of the relatively denser warmer water. Above 4^{o}C the warming of the upper layers leads to an increasingly pronounced stable density stratification and a correspondingly increased resistance of the water to vertical mixing, so that nearly all of the downward distribution of heat through the lake must be accomplished during periods of wind sufficiently strong to perform the required work against gravity. Birge (1915) has defined the winter heat budget of a temperate lake as the gain in heat content from the minimum value to the value for an isothermal state at 4°C, and his definition of the summer heat budget is analogous.

In this paragraph, the interaction between meteorological variables and a temperate lake during the open season will be described briefly. If one assumes a balance between net radiation, sensible heat flux, latent heat flux and the time derivative of heat content as do Bryson and Dutton (1960) in their study of Lake Mendota, then the evaporative flux is expressible as a function of net radiation, change of heat storage and the Bowen ratio. Net radiation depends on incoming solar radiation, albedo of the lake surface and the long-wave back radiation, which in turn varies with air temperature, relative humidity and cloudiness. During the open season, according to Bryson and Dutton (1960) the heat content curve for Lake Mendota roughly follows the net radiation curve in gross from with a lag of about a month and a half. The relatively large values of net radiation and low evaporation in spring contribute to the rapid increase of heat content soon after the opening of the lake. While the thermal stratification is still relatively weak, periods of strong winds also contribute to heat content increases by mixing the water.

In the summer the relatively high temperature and humidity of the air create a positive feedback by reducing back radiation. The time of maximum heat content usually follows the time of maximum surface temperature by about two weeks, since the lowering of the thermocline increases the thickness of the warm layer above. On the other hand, after the disappearance of the thermocline in autumn, the much lower net radiation and relatively high evaporation enhanced by a positive water-air temperature gradient and windy periods encourage rapid loss of heat content. The decreasing temperature and humidity of the air make back radiation more effective, again producing a positive feedback.

In general, neither T nor w can be expressed as a simple closed function of depth on the basis of temperature soundings and hypsometric data. To compute heat content, Bryson and Dutton (1960) use finite sums to approximate the integral in equation (1), dividing a lake into one-meter slices with representative values of T and w assigned to each slice. Their method has the advantage that it can be used to calculate heat fluxes not only through an entire lake but also through the individual layers; however, the deeper the lake basin the more calculations are required to compute θ.

This paper will describe a short approximate method such that the number of computations required is independent of the depth of the lake, using simple closed functions of depth to approximate w and T. A power curve will be fitted to the basin so that the volume development as defined by Hutchinson (1957) will be equal for the real basin and the approximating basin. From any given sounding the lake will be partitioned systematically into two layers with one representative temperature assigned to each layer. An empirical method of correcting for possible large errors in the resulting heat content value will be described, based on the differences in shape between the true sounding and the two-layer approximation. Using the values obtained from the slice method for comparison, the relative error of the short method with and without the correction for 46 actual open-season soundings from nine lakes will be discussed. The closeness of fit of the smooth curves for w(z) to the hypsometric data will also be examined.

2. Description of the Method

For a lake of maximum depth H and surface area A_o, let z be the height above the lowest point of the bottom, so that z = H at the surface. Let A(z) be the cross-sectional area for any value of z, so that $A(H) = A_o$. The normalized dimensionless variables h and w are defined by

$$h = z/H \qquad (2)$$

$$w = A/A_o \qquad (3)$$

so that w becomes the area weighting factor for heat content calculations. Equation (1) then becomes

$$\theta = H\int_0^1 cT(h)w(h)\,dh \tag{4}$$

by the change of variable z = Hh. Always w = 0 when h = 0, and w = 1 when h = 1, while w is a monotonic increasing function. In practice, w is known from bathymetric measurement as a function of h, but cannot be written exactly as any simple closed function of h. However, if w is replaced by a simple power function

$$w' = h^p \tag{5}$$

where p is a positive number, then clearly w' has all the properties of w just mentioned. In practice, a curve described by equation (5) is found which best fits the hypsometric profile of a lake.

Regard the lake as having unit surface area and unit maximum depth. The volume of the normalized lake is a number between 0 and 1, approaching zero for highly convex lake basins and approaching unity for concave basins which are close to U-shaped. The lake basin is fitted to a solid of revolution having the same normalized volume as the real basin. A solid with cross-sectional area h^p is generated by revolving the curve $f(h) = \frac{1}{\sqrt{\pi}}h^{p/2}$ about the h-axis from h = 0 to h = 1. The volume as a function of p is

$$V(p) = 1/(1+p) \tag{6}$$

by straightforward integration. In particular, p = 1 and p = 2 represent a paraboloidal basin and a conical basin respectively. Values of p greater than 2 correspond to convex basins, and values of p approaching zero correspond to basins that are almost U-shaped. To obtain a normalized volume V_h from N given hypsometric data points (h_i, w_i) arranged in order of increasing values of the variables, let $\Delta h_i = h_{i+1} - h_i$ and $\bar{w}_i = (w_{i+1} + w_i)/2$, taking the average value of w in the k<u>th</u> slice (k = 1, 2, ..., N-1) for computational convenience. By numerical integration,

$$V_h = \sum_{i=1}^{N-1} \bar{w}_i \Delta h_i \tag{7}$$

It is found by solving equation (6) for p with $V(p) = V_h$ that the corresponding exponent p must be

$$p = (1/V_h) - 1 \tag{8}$$

so that once V_h has been computed, a curve is uniquely determined by a quick substitution into equation (8). Hutchinson (1957) defines the volume development D of a lake basin as the ratio of the actual basin volume to that of a cone having height H and base area A_o, so that $D = 3V_h = 3/(1+p)$. The curve from (8) replaces the actual hypsometric profile of a lake basin by a simple power-law profile such that the volume developments of the real basin and of its idealized counterpart are equal. By the manner in which the curve fit is arranged, correct heat contents are guaranteed for any isothermal sounding, neglecting the slight error arising from measurement and numerical integration. In such a case $T(h) = T_o$ where T_o is constant, and by equation (4) the heat content is

$$\theta = cHT_o \int_0^1 w(h)\,dh \qquad (9)$$

The integral is simply V_h, which by the definition of p is the same number obtained by replacing w by $w' = h^p$.

Generally, neither T nor w has a convenient closed form as a function of depth. The method to be described substitutes a simple closed form using a given sounding as a basis but making use of only five temperatures regardless of the sounding. Only two of these measurements are used for the computation. The actual hypsometric profile is replaced by its power-law fit.

Lake soundings which exhibit pronounced summer stratification approximate an idealized step-function form given by

$$T'(z) = \begin{cases} T_2 & \text{if} \quad z^* < z \leq H \\ T_1 & \text{if} \quad 0 \leq z < z^* \end{cases} \qquad (10)$$

The simplicity of this ideal temperature curve motivates the technique used in obtaining two representative temperatures from a given sounding for use in estimating heat content, regardless of the number of data points used to construct the original sounding. Let T_t and T_b denote the surface temperature and the temperature at the deepest point of the lake respectively. Define $\bar{T}$ to be the arithmetic mean of T_t and T_b, and let d* denote the depth in centimeters below the surface such that $T = \bar{T}$. Assume for concreteness that T_t is higher than T_b and that the temperature increases monotonically with h, so that the value $\bar{T}$ occurs at only one depth. (In the uncommon event that $\bar{T}$ be taken on more than once or throughout an interval, the highest level at which $\bar{T}$ occurs can be used for definiteness.) Let z* = H - d*, and let h* = z*/H. This level partitions a lake into an upper layer and a lower layer. Generally, T is not constant in either layer, but from the appearance of many soundings it looks reasonable to take for T_1 the temperature at the middle of the lower layer, where z = z*/2 and h = h*/2, and to take for T_2 the temperature at the middle of the upper layer, where z* = (H+z*)/2 and h = (1+h*)/2.

In practice, when the temperature is recorded at depths which are usually equidistant, d* (hence z* and h*), T_1 and T_2 can be found conveniently by linear interpolation between the data points. Five temperatures -- T_t, T_b, $\bar{T}$, T_1 and T_2 -- have been used in all, and the lake is treated as if it consisted of an isothermal upper layer of temperature T_2 and an isothermal lower layer of temperature T_1. It must be noted that the two-layer temperature approximation is an artificial model that has been constructed mainly for simplicity. While a temperature curve exhibiting very sharp stable stratification and a narrow thermocline approximates the step function reasonably well, the thermocline can be diffuse even with a large range in temperature, and when T is only slightly higher at the surface than at the bottom the temperature curve tends to be linear.

Let Q denote the heat content obtained by replacing w by $w' = h^p$ and T by T' for a lake of normalized volume V_h, in contrast to the actual heat content θ. Then by equation (4),

$$Q = cHT_1 \int_0^{h^*} h^p dh + cHT_2 \int_{h^*}^{1} h^p dh \qquad (11)$$

By direct integration and some rearranging of terms,

$$Q = cV_h \left[HT_2 - (h^*)^p z^* \, (T_2 - T_1) \right] \qquad (12)$$

From equation (12), it is apparent that the more nearly isothermal the lake, the more closely Q estimates the actual heat content θ, for then Q approaches cV_hHT_2, the true heat content of a lake with normalized volume V_h and uniform temperature T_2. For an isothermal sounding, z* (and hence h*) is arbitrary, but this fact results in no real difficulty since no computation is necessary beyond that of V_h.

Figure 1 is a schematic diagram of strong summer stratification with the two-layer approximating temperature curve superimposed on the sounding. If the direction of increasing temperature is taken to be upward, the schematic sounding is approximately linear in the upper half of the upper layer, concave downward in the lower half of the upper

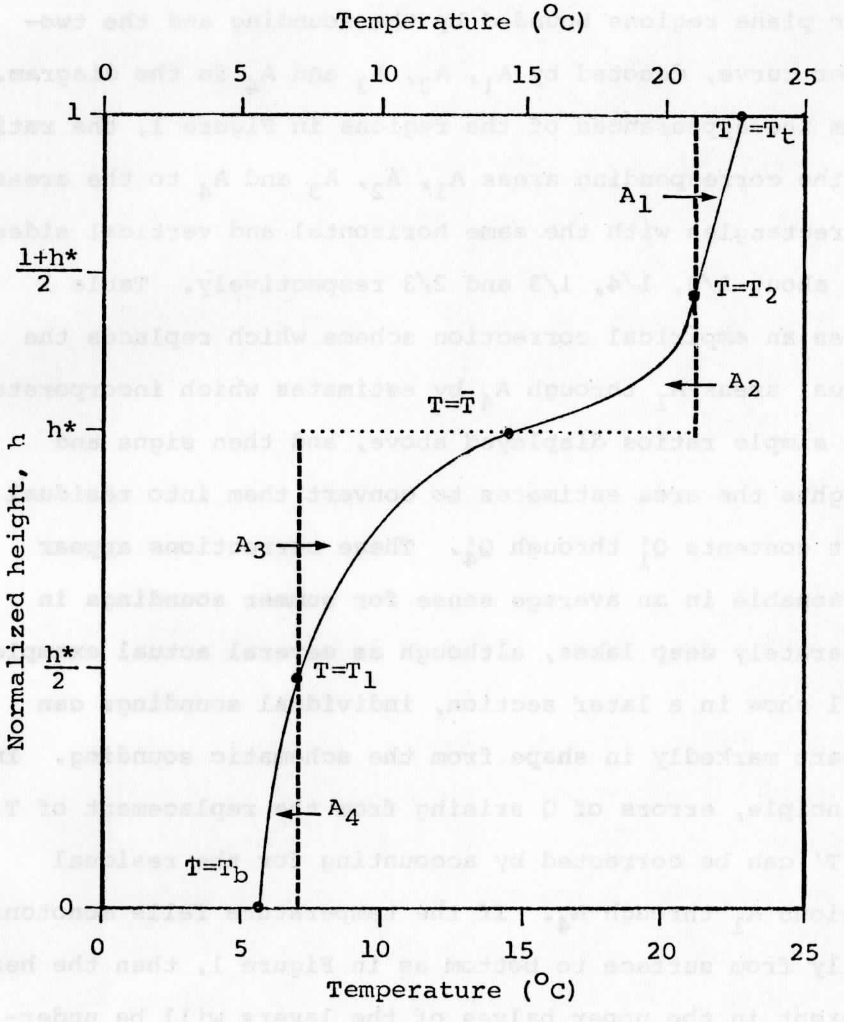

Figure 1. Schematic diagram of pronounced summer stratification (solid curve) with superimposed two-layer approximation (dotted curve) and residual areas bounded by the two curves.

layer and concave upward in the lower layer. There are four plane regions bounded by the sounding and the two-layer curve, denoted by A_1, A_2, A_3 and A_4 in the diagram. From the appearances of the regions in Figure 1, the ratios of the corresponding areas A_1, A_2, A_3 and A_4 to the areas of rectangles with the same horizontal and vertical sides are about 1/2, 1/4, 1/3 and 2/3 respectively. Table 1 gives an empirical correction scheme which replaces the actual areas A_1 through A_4 by estimates which incorporate the simple ratios displayed above, and then signs and weights the area estimates to convert them into residual heat contents Q_1' through Q_4'. These corrections appear reasonable in an average sense for summer soundings in moderately deep lakes, although as several actual examples will show in a later section, individual soundings can depart markedly in shape from the schematic sounding. In principle, errors of Q arising from the replacement of T by T' can be corrected by accounting for the residual regions A_1 through A_4. If the temperature falls monotonically from surface to bottom as in Figure 1, then the heat content in the upper halves of the layers will be underestimated while the heat content in the lower halves will be overestimated.

The heat content of any layer within a lake can be found by applying equation (1) or equation (4) with an appropriate change of the limits of integration, and Q_1' through Q_4' are computed by approximating to this process as shown in Table 1. The weighting factors for each of the four regions are calculated at their vertical midpoints from the power curve in order to avoid tedious interpolation between hypsometric data points. The corrected heat content is

$$Q' = Q + Q_1' + Q_2' + Q_3' + Q_4' \qquad (13)$$

The empirical correction method demonstrated in Table 1 treats all soundings as if the temperature curve had the same shape in corresponding half-layers. This simplifying assumption is not always realistic, since an open-season sounding can deviate markedly in shape from the schematic curve of Figure 1. However, a sounding with a poorly defined or nonexistent thermocline generally has a considerably smaller range of temperature than a strongly stratified sounding, and both the actual and the estimated heat content differences become smaller in comparison to any of θ, Q or Q' for given values of H, h* and p as an isothermal state is approached. Moreover, heat content is an integral, and signed areas which cancel at least partially are being added up. It is then reasonable to expect acceptable accuracy for Q' even when the temperature range is large and one or more of the residual regions is not closely estimated,

TABLE 1

CORRECTION METHOD BY RESIDUAL AREAS

	Upper layer, upper half (region A_1)	Upper layer, lower half (region A_2)	Lower layer, upper half (region A_3)	Lower layer, lower half (region A_4)
(1)	$x_1=T_t-T_2$	$x_2=T_2-\bar{T}$	$x_3=\bar{T}-T_1$	$x_4=T_1-T_b$
(2)	$y_1=(1-h^*)/2$	$y_2=(1-h^*)/2$	$y_3=h^*/2$	$y_4=h^*/2$
(3)	$A_1'=x_1y_1/2$	$A_2'=x_2y_2/4$	$A_3'=x_3y_3/3$	$A_4'=2x_4y_4/3$
(4)	$h_1=1-(1-h^*)/4$	$h_2=1-3(1-h^*)/4$	$h_3=3h^*/4$	$h_4=h^*/4$
(5)	$W_1=h_1^p$	$W_2=h_2^p$	$W_3=h_3^p$	$W_4=h_4^p$
(6)	$Q_1'=cHW_1A_1'$	$Q_2'=-cHW_2A_2'$	$Q_3'=cHW_3A_3'$	$Q_4'=-cHW_4A_4'$

x_k - Length of horizontal straight side of A_k (^oC)

y_k - Length of vertical straight side of A_k

A_k' - Approximate area of A_k, based on schematic sounding in Figure 1 (^oC)

h_k - Value of h at vertical midpoint of A_k

W_k - Weighting factor for A_k

Q_k' - Approximate heat content correction corresponding to A_k (cal cm^{-2})

provided that the error committed in fitting the hypsometric profile of the lake to the power curve is small. It will be seen in a later section that the heat content obtained by retaining the actual temperature curve and replacing w by w' differs only slightly from the Bryson-Dutton heat content for a variety of lake shapes and soundings. In the few cases of large discrepancy of Q, nearly all of the difference results from approximating the temperature by the step function.

3. Illustrative Examples

This section will illuminate the preceding description by presenting an example of a hypsometric volume computation and by applying the two-layer temperature approximation with the correction scheme to a particular summer sounding.

In Table 2, p is calculated for Rocky Lake, Manitoba, which has a maximum depth of 8 meters. Temperatures are taken at one-meter intervals from the surface downward, so that there are N = 9 hypsometric data points and $\Delta h_i = .125$ for i = 1, 2, ..., 8. Since $V_h = .584$ and p = .712, the basin of Rocky Lake is somewhat flatter than a paraboloid and has a volume development of D = 3 x .584 = 1.75, indicating that Rocky Lake has a volume 1.75 times that of a conical basin having the same surface area and total depth.

TABLE 2

CALCULATION OF HYPSOMETRIC VOLUME FOR ROCKY LAKE

i	h_i	w_i	$\bar{w}_i$	Δh_i	$\bar{w}_i \Delta h_i$
1	.000	.000	.080	.125	.0100
2	.125	.161	.249	.125	.0311
3	.250	.337	.414	.125	.0518
4	.375	.490	.554	.125	.0692
5	.500	.618	.679	.125	.0849
6	.625	.740	.813	.125	.1016
7	.750	.886	.913	.125	.1141
8	.875	.940	.970	.125	.1212
9	1.000	1.000	---	---	----

$$V_h = \sum_{i=1}^{8} \bar{w}_i \Delta h_i = .5839 = .584 \text{ (approx.)}$$

$$p = (1/V_h) - 1 = .712$$

A sounding taken on September 7, 1960 at Tub Lake, Wisconsin is shown in Figure 2, with the two-layer approximating temperature curve superimposed on the sounding. Temperatures have been taken at one-meter intervals from the surface to the bottom at a depth of 8 meters, and the data points have been connected by linear segments so that $\bar{T}$, T_1 and T_2 can be estimated by linear interpolation. In this case T_t=25.1°C and T_b = 5.6°C, so that $\bar{T}$ = 15.4°C. The temperatures at depths of 2 and 3 meters are 19.0°C and 15.0°C respectively. By interpolation d* = 2.9 m so that z* = 510 cm and h* = z*/H = .636, and T = T_1 = 6.3°C at a depth of 5.45 m. Likewise, T = T_2 = 20.4°C at a depth of 1.45 m.

Despite a very large temperature contrast between surface and bottom, the stratification exhibited in Figure 2 is not sharp, and the two-layer step function is not a good approximation to the true temperature curve. The thermocline is diffuse, extending roughly from the surface to 4 meters with a fairly steady temperature fall averaging close to 4°C per meter. As a result, there is no well defined epilimnion although there is a well-defined hypolimnion below 5 meters, in contrast to the ideal hypolimnion which begins at 2.9 meters. The sounding departs in shape from the schematic sounding of Figure 1 due to the linearity in the upper layer extending throughout the layer, and the

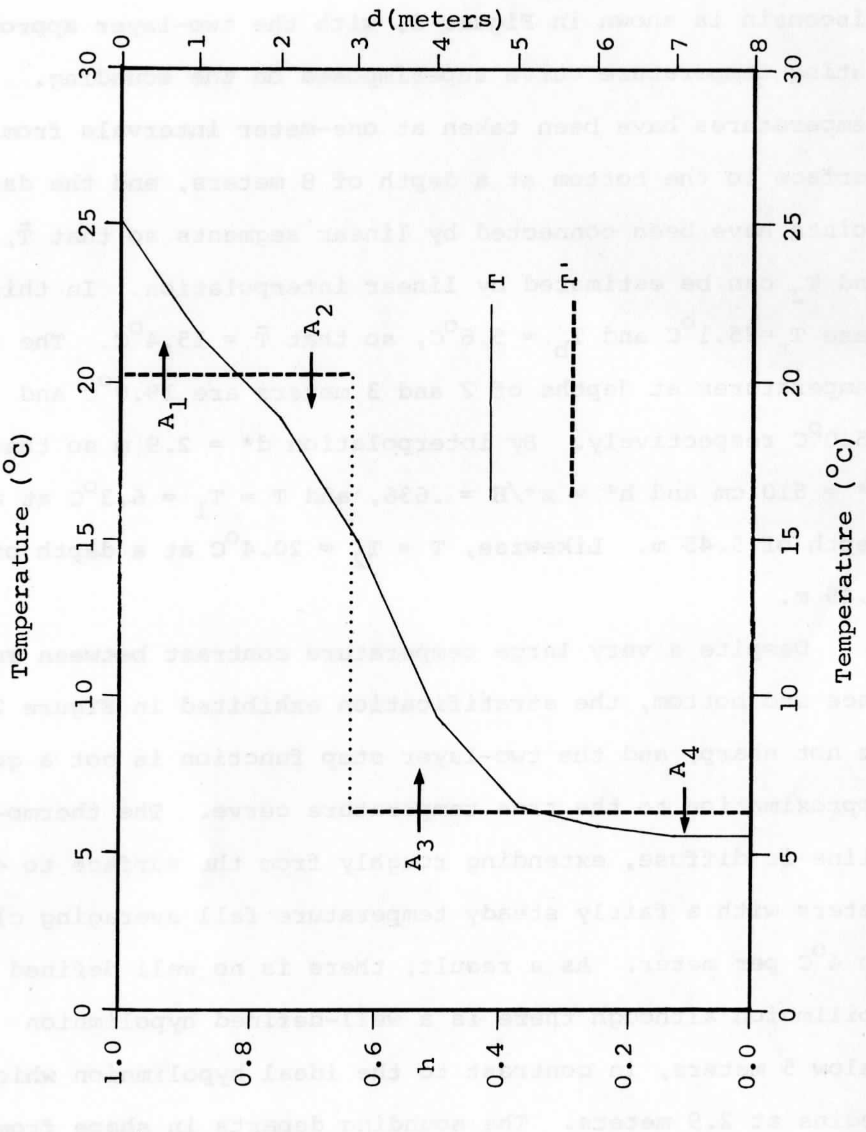

Figure 2. Tub Lake sounding of September 7, 1960.

TABLE 3

CORRECTION METHOD APPLIED TO
TUB LAKE SOUNDING OF SEPT. 7, 1960

	Upper layer, upper half	Upper layer, lower half	Lower layer, upper half	Lower layer, lower half
(1)	x_1=4.7	x_2=5.0	x_3=9.1	x_4=0.7
(2)	y_1=.182	y_2=.182	y_3=.318	y_4=.318
(3)	A_1'=.428 (.396*)	A_2'=.228 (.413*)	A_3'=.964 (1.072*)	A_4'=.149 (.164*)
(4)	h_1=.909	h_2=.727	h_3=.477	h_4=.159
(5)	$W_1=.909^{1.209}$ =.891	$W_2=.727^{1.209}$ =.680	$W_3=.477^{1.209}$ =.409	$W_4=.159^{1.209}$ =.109
(6)	Q_1'=305	Q_2'=-124	Q_3'=315	Q_4'=-13

$$Q = 5.504 \times 10^3 \text{ cal cm}^{-2}$$

$$Q' = Q + Q_1' + Q_2' + Q_3' + Q_4'$$
$$= 5504 + 305 - 124 + 315 - 13$$
$$= 5.987 \times 10^3 \text{ cal cm}^{-2}$$

$$\theta = 5.961 \times 10^3 \text{ cal cm}^{-2}$$

*Actual areas given for comparison.

typical radical increase in temperature gradient as h* is approached does not occur. Calculations of Q and Q', together with the step-by-step computations of the empirical correction method, appear in Table 3. The shapes of the regions A_1, A_3 and A_4 are reasonably well estimated, but the area A_2 is notably underestimated due to the linearity of the sounding.

From this point onward, the symbol θ will refer to heat content values obtained by the Bryson-Dutton method rather than to the exact heat content defined by equation (1) or (4). In the present example, Q is about seven percent smaller than θ, chiefly because of the large discrepancy between the actual and approximating temperature curves in the upper half of the lower layer: T' underestimates T by an average value of about $3^{\circ}C$, with an error of $9^{\circ}C$ just below the level h*. In this example, the correction scheme effects a dramatic improvement since Q' is within one percent of θ.

4. The Data

The short approximate method of calculating heat content described in the previous section was tested on 46 open-season soundings from nine temperate lakes, four in the United States and five in Canada, ranging in depth from 8 to 35 meters. Table 4 lists the lakes along with their locations, depths and the number of soundings tested for each

TABLE 4

LIST OF LAKES USED

Lake and location	H (m)	Number of soundings
American:		
Lake Mendota (Wisconsin)	23	9
Tub Lake (Wisconsin)	8	8
Trout Lake (Wisconsin)	35	6
Deer Lake (Minnesota)	33	6
Canadian:		
Rocky Lake (Manitoba)	8	2
Clear Lake (Manitoba)	33	4
Zed Lake (Manitoba)	25	2
Waskesiu Lake (Saskatchewan)	21	6
Grace Lake (Yukon)	17	3

TABLE 5

HYPSOMETRIC DATA

Lake	V_h	p
Rocky	.584	.712
Mendota	.535	.870
Waskesiu	.522	.915
Tub	.452	1.209
Clear	.413	1.416
Deer	.404	1.476
Trout	.393	1.544
Grace	.348	1.874
Zed	.298	2.356

lake.

For all lakes other than Lake Mendota, soundings and hypsometric measurements were taken in 1958, 1959, 1960 or 1961 by members of the Department of Meteorology at the University of Wisconsin. On each date, soundings were taken at ten to thirty points and representative temperatures at each whole-meter depth were obtained by averaging the individual temperatures at that depth. Maximum depths were estimated to the nearest meter from the hypsographs. In the few cases in which a sounding did not go to the maximum depth, the temperature at the deepest level reached was repeated down to the maximum depth for purposes of heat content computation.

The soundings for Lake Mendota, dating from 1897 to 1914, were taken from the unpublished data of E. A. Birge (University of Wisconsin Archives), who took single soundings on each date. The hypsometric weighting factors for Lake Mendota at one-meter intervals from the surface to 23 meters were computed by linear interpolation from values listed by Bryson and Dutton (1960) at five-foot intervals from the surface to 75 feet. The Birge soundings extend to 23 meters, where only 1.0% of the surface area remains.

5. Results

Table 5 lists H, V_h and p for the nine lakes used in this study. The surface areas and fetches of the lakes are not included, since the importance of these variables is not being considered and heat content is evaluated per square centimeter only. Three of the lakes (Rocky, Mendota and Waskesiu) are somewhat flatter than paraboloidal and five lakes (Tub, Clear, Deer, Trout and Grace) have basins intermediate between paraboloidal and conical, while one lake (Zed) is slightly convex upward. The hypsometric data points and the best fitting power curves are shown in Figures 3-5.

For four of the lakes (Mendota, Clear, Grace and Zed) there is a tendency for w' to underestimate w in the lower layers and to overestimate w in the upper layers. The fits for Tub and Deer are nearly perfect in the lower halves of their basins, but w is underestimated between h = .5 and h = .8 and overestimated in the uppermost levels. By contrast, w is overestimated in the lower half of the Rocky Lake basin and underestimated throughout the upper half. The magnitude of the deviations is generally between .02 and .04, but positive deviations of .07 occur near h = .9 for Deer Lake and Zed Lake. The curve fit for Rocky Lake shows deviations up to .07 in both directions. The overestimation of w persists over a large interval of the Grace Lake basin, from about h = .4 to the surface. The curve fits for Waskesiu Lake and Trout Lake are almost perfect, since w' is within .02 of w at all data points and most of

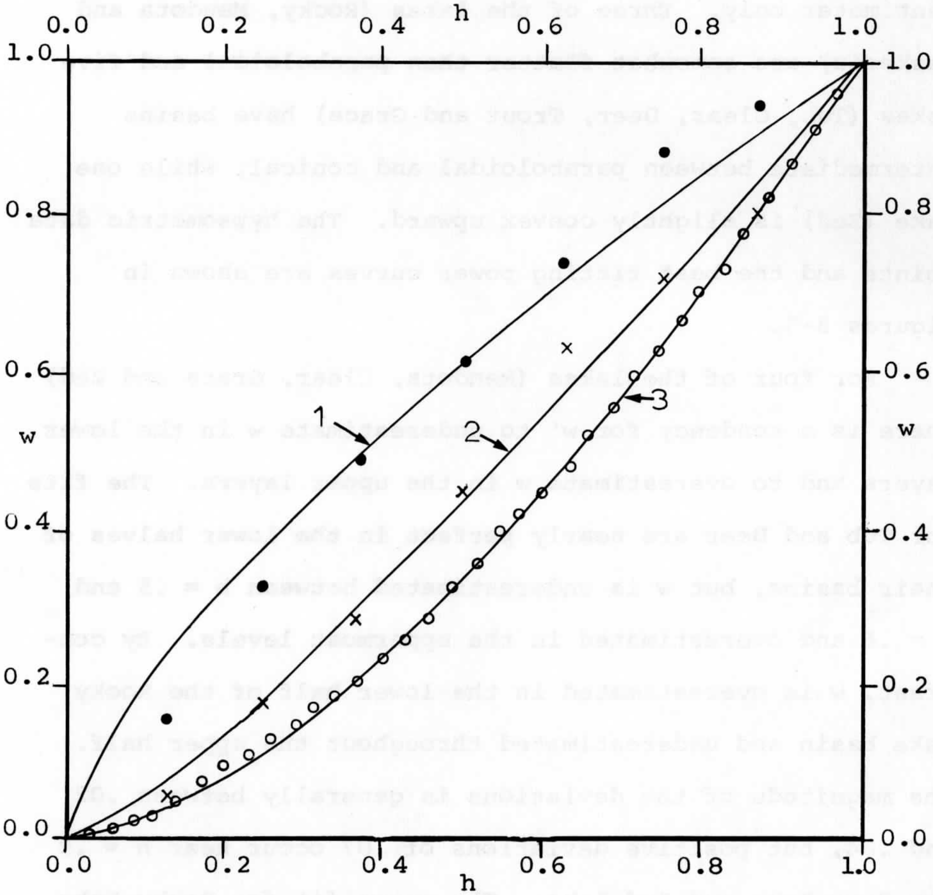

Figure 3. Hypsometric data points and approximating power curves for Rocky Lake (curve #1, filled circles), Tub Lake (curve #2, crosses) and Trout Lake (curve #3, open circles).

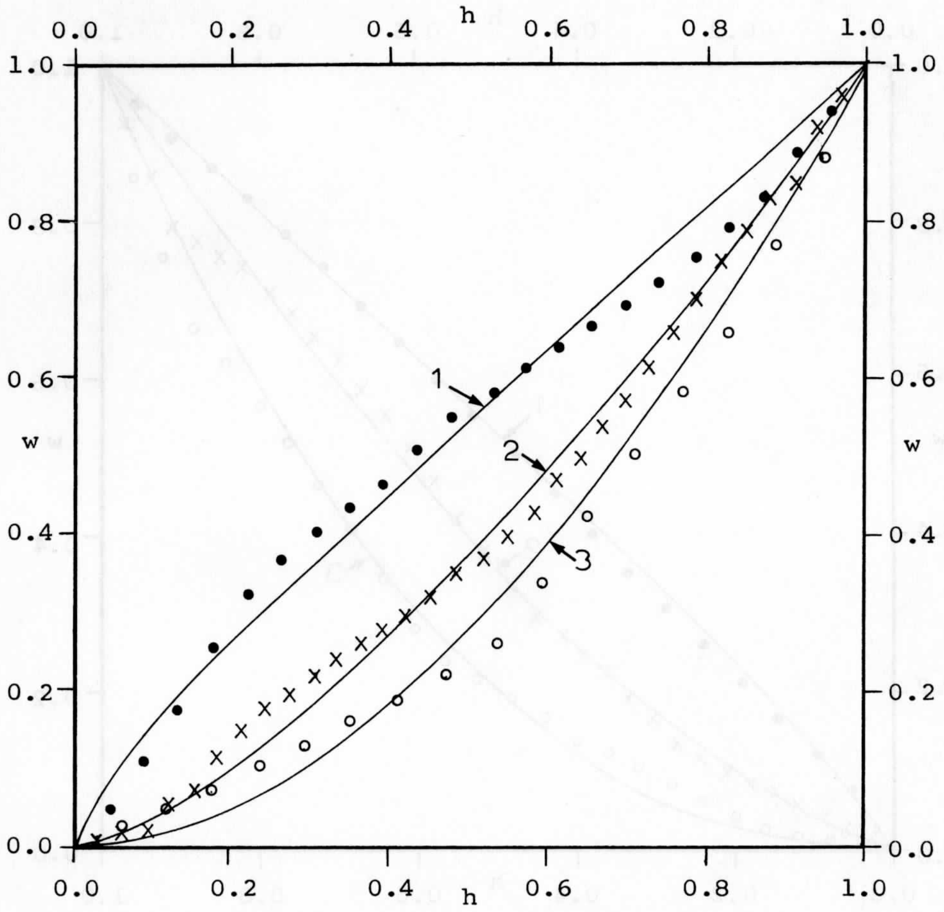

Figure 4. Hypsometric data points and approximating power curves for Lake Mendota (curve #1, filled circles), Clear Lake (curve #2, crosses) and Grace Lake (curve #3, open circles).

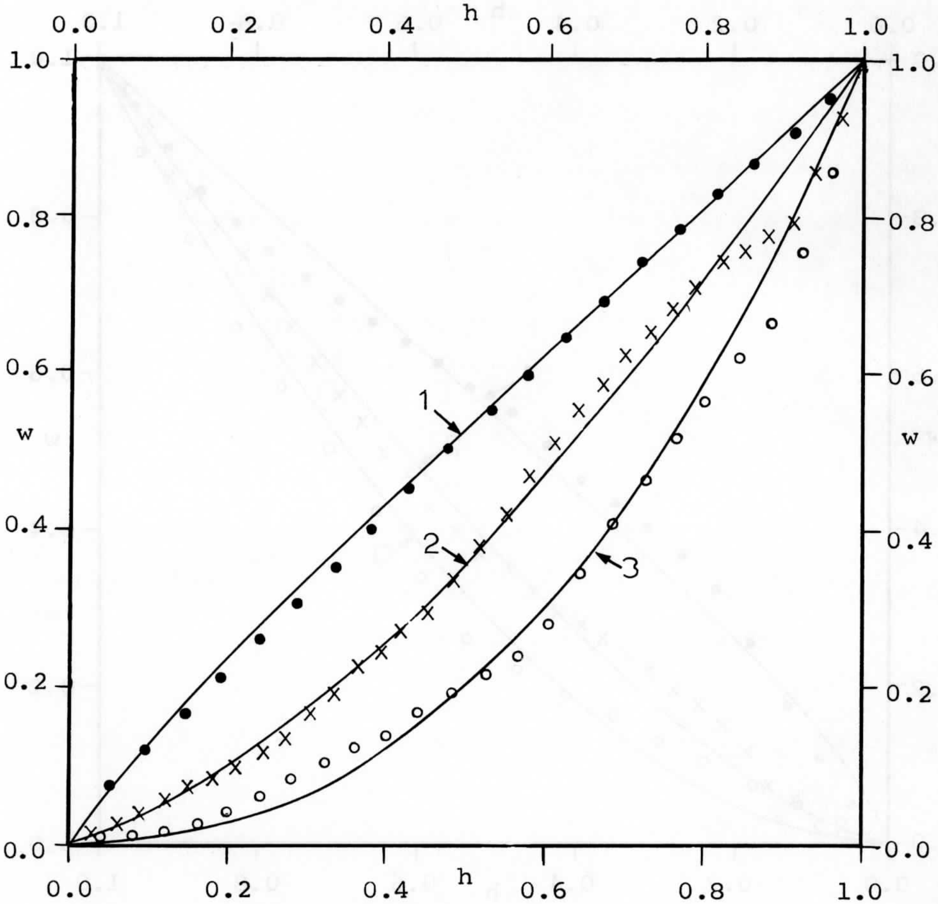

Figure 5. Hypsometric data points and approximating power curves for Waskesiu Lake (curve #1, filled circles), Deer Lake (curve #2, crosses) and Zed Lake (curve #3, open circles).

the points cannot be separated visually from the power curve.

For each of the soundings tested, Θ was computed by the slice method of Bryson and Dutton (1960) with one minor modification: the one-meter layers were taken between consecutive whole-meter levels and the average of the known end-point temperatures was used as a representative temperature in each slice. (In their analysis of heat flux in Lake Mendota, Bryson and Dutton center the slices at the whole-meter levels and use the temperatures at these levels.) $\bar{T}$ was determined from the surface and bottom temperatures; h*, T_1 and T_2 were found by linear interpolation. Q was calculated from equation (12) and Q' by the correction scheme outlined in Table 1. The temperatures and values of h* appear in Table 6(a), and the heat content figures with the percentage differences between Q or Q' and Θ are listed in Table 6(b).

Before discussing the differences, it is desirable to mention two main sources of uncertainty in Θ itself. First, except near the bottom, the temperatures at each meter level are averages of 10 to 30 measurements with an associated probable error. Bryson and Dutton (1960) attach a probable error of $0.1^{\circ}C$ to $0.5^{\circ}C$ to the averages they calculate in their analysis of Lake Mendota. Second, for a summer sounding which features a large and variable temperature gradient in a one-meter slice, the arithmetic mean of the

TABLE 6(a)

CRITICAL TEMPERATURES AND h* FOR TEST SOUNDINGS

Sounding		T_t °C	T_2 °C	$\bar{T}$ °C	T_1 °C	T_b °C	h*
Rocky Lake	5/27/58	12.1	12.0	11.6	11.0	11.2	.271n
	7/29/59	22.5	22.0	21.7	20.9	20.8	.328n
Tub Lake	7/12/58	26.6	22.4	16.2	6.6	5.8	.688
	8/ 7/58	29.0	25.7	17.5	6.6	6.0	.662
	9/ 3/58	19.5	18.5	13.0	7.0	6.4	.556
	6/12/59	24.3	22.8	14.6	5.4	5.0	.698
	7/17/59	25.4	23.6	15.5	6.6	5.6	.611
	8/18/59	22.7	21.8	14.4	6.8	6.1	.556
	6/14/60	26.3	21.7	15.8	5.6	5.2	.804
	9/ 7/60	25.1	20.4	15.4	6.3	5.6	.636
Zed Lake	7/17/58	16.9	14.8	12.7	10.3	8.5	.520n
	8/ 2/59	17.7	17.2	13.3	10.3	8.8	.371
Clear Lake	8/21/58	20.6	19.8	16.9	14.3	13.2	.417
	5/25/59	6.8	6.6	6.1	5.9	5.4	.485n
	7/25/59	20.3	19.8	15.5	11.2	10.6	.618
	8/18/60	20.2	20.2	15.8	11.5	11.4	.587

n - The value of h* cannot be regarded as representing a thermocline level, since no definite thermocline is apparent.

TABLE 6(a)

CRITICAL TEMPERATURES AND h* FOR TEST SOUNDINGS (continued)

Sounding		T_t °C	T_2 °C	$\overline{T}$ °C	T_1 °C	T_b °C	h*
Waskesiu Lake	7/21/59	20.7	19.1	16.2	13.0	11.7	.593
	5/28/60	8.0	7.5	6.7	5.3	5.3	.579n
	6/22/60	14.7	12.8	11.3	9.4	7.8	.595n
	7/ 9/60	19.0	18.0	14.6	11.8	10.1	.664
	7/18/60	21.8	20.3	16.2	12.3	10.5	.578
	8/ 9/60	20.3	18.9	16.3	14.9	12.4	.341
Lake Mendota	6/30/97	21.4	21.4	16.5	12.1	11.7	.585
	8/ 5/98	22.4	22.0	16.4	11.3	10.4	.565
	7/ 9/06	21.4	20.5	16.3	12.1	11.1	.614
	7/30/06	25.5	24.7	19.0	13.5	12.5	.622
	7/ 5/11	26.0	25.6	18.4	11.5	10.7	.612
	6/26/12	21.8	18.6	16.5	12.7	11.3	.549n
	7/31/12	23.6	22.5	18.0	13.7	12.4	.585
	7/18/14	24.4	24.1	17.7	11.9	11.0	.565
	9/ 7/14	20.6	20.4	16.2	12.7	11.7	.441

n - The value of h* cannot be regarded as representing a thermocline level, since no definite thermocline is apparent.

TABLE 6(a)

CRITICAL TEMPERATURES AND h* FOR TEST SOUNDINGS (continued)

Sounding		T_t °C	T_2 °C	$\overline{\overline{T}}$ °C	T_1 °C	T_b °C	h*
Grace Lake	8/19/59	14.0	13.3	9.2	5.5	4.4	.615
	7/27/60	17.5	17.6	11.4	6.3	5.3	.651
	6/27/61	19.8	17.3	12.5	6.4	5.2	.706
Trout Lake	5/30/58	11.9	10.8	8.5	6.4	5.1	.581n
	5/19/59	9.8	8.6	8.3	7.7	6.7	.700n
	7/ 3/59	19.1	19.1	13.2	8.1	7.2	.697
	8/25/59	23.1	22.5	15.3	8.4	7.5	.669
	7/ 5/60	18.6	18.2	12.0	5.9	5.4	.736
	8/17/61	22.4	21.8	14.1	6.8	5.9	.717
Deer Lake	9/ 9/58	18.3	18.1	13.5	9.6	8.7	.565
	5/ 4/59	7.4	6.7	6.3	5.3	5.1	.803n
	7/20/59	23.0	22.0	15.6	9.5	8.1	.652
	8/24/59	24.2	22.0	16.8	9.5	9.3	.656
	9/26/59	16.1	16.1	12.1	9.1	8.0	.493
	8/19/60	21.9	21.8	14.3	7.7	6.6	.689

n - The value of h* cannot be regarded as representing a thermocline level, since no definite thermocline is apparent.

TABLE 6(b)

CALCULATED HEAT CONTENTS
FOR TEST SOUNDINGS

	Sounding	Θ, 10^4 cal cm^{-2}	Q, 10^4 cal cm^{-2}	Diff., %	Q', 10^4 cal cm^{-2}	Diff., %
Rocky Lake	5/27/58	.559	.556	-0.6	.556	-0.4
	7/29/59	1.027	1.022	-0.5	1.027	-0.1
Tub Lake	7/12/58	.603	.559	-7.3	.608	+0.7
	8/ 7/58	.694	.653	-5.9	.694	-0.1
	9/ 3/58	.560	.555	-0.8	.562	+0.4
	6/12/59	.562	.542	-3.6	.570	+1.4
	7/17/59	.661	.646	-2.4	.664	+0.5
	8/18/59	.641	.642	+0.1	.647	+0.9
	6/14/60	.485	.426	-12.2	.491	+1.3
	9/ 7/60	.596	.550	-7.7	.599	+0.4
Zed Lake	7/17/58	1.095	1.065	-2.7	1.106	+1.0
	8/ 2/59	1.247	1.263	+1.3	1.261	+1.1
Clear Lake	8/21/58	2.596	2.611	+0.6	2.621	+1.0
	5/25/59	.882	.879	-0.4	.883	+0.1
	7/25/59	2.343	2.339	-0.2	2.356	+0.5
	8/18/60	2.424	2.432	+0.3	2.430	+0.2

TABLE 6(b)
CALCULATED HEAT CONTENTS
FOR TEST SOUNDINGS
(continued)

	Sounding	Θ, 10^4 cal cm^{-2}	Q, 10^4 cal cm^{-2}	Diff., %	Q', 10^4 cal cm^{-2}	Diff., %
Waskesiu Lake	7/21/59	1.882	1.848	-1.8	1.879	-0.1
	5/28/60	.754	.739	-2.0	.754	+0.1
	6/22/60	1.305	1.268	-2.9	1.298	-0.5
	7/ 9/60	1.675	1.665	-0.6	1.676	+0.1
	7/18/60	1.947	1.918	-1.4	1.940	-0.3
	8/ 9/60	2.031	2.019	-0.6	2.034	+0.2
Lake Mendota	6/30/97	2.191	2.211	+0.9	2.215	+1.1
	8/ 5/98	2.280	2.253	-1.2	2.258	-1.0
	7/ 9/06	2.134	2.105	-1.3	2.130	-0.2
	7/30/06	2.489	2.471	-0.7	2.499	+0.4
	7/ 5/11	2.442	2.458	+0.7	2.480	+1.6
	6/26/12	2.136	2.056	-3.7	2.137	+0.1
	7/31/12	2.380	2.371	-0.3	2.392	+0.5
	7/18/14	2.402	2.448	+1.9	2.450	+2.0
	9/ 7/14	2.297	2.305	+0.3	2.285	-0.5

TABLE 6(b)

CALCULATED HEAT CONTENTS
FOR TEST SOUNDINGS
(continued)

	Sounding	Θ, 10^4 cal cm^{-2}	Q, 10^4 cal cm^{-2}	Diff., %	Q', 10^4 cal cm^{-2}	Diff., %
Grace Lake	8/19/59	.676	.674	-0.2	.679	+0.5
	7/27/60	.818	.846	+3.4	.842	+2.9
	6/27/61	.818	.786	-3.8	.830	+1.5
Trout Lake	5/30/58	1.337	1.331	-0.4	1.358	+1.6
	5/19/59	1.163	1.131	-2.7	1.160	-0.3
	7/ 3/59	2.037	2.023	-0.7	2.041	+0.2
	8/25/59	2.432	2.401	-1.3	2.436	+0.2
	7/ 5/60	1.762	1.730	-1.8	1.788	+1.5
	8/17/61	2.152	2.112	-1.9	2.173	+1.0
Deer Lake	9/ 9/58	2.141	2.135	-0.3	2.127	-0.7
	5/ 4/59	.812	.783	-3.5	.810	-0.2
	7/20/59	2.364	2.352	-0.5	2.387	+1.0
	8/24/59	2.415	2.342	-3.0	2.441	+1.1
	9/26/59	1.984	1.983	-0.1	1.957	-1.4
	8/19/60	2.188	2.159	-1.3	2.183	-0.2

top and bottom temperatures in such a layer may be 1°C to 2°C too high or low to be representative.

Consider a hypothetical lake with a parabolic basin 25 meters deep, and suppose that the layer from 7 to 8 meters (so that $\bar{w}$ = .7) has top and bottom temperatures of 20°C and 14°C, with an average temperature of only 15°C because of a highly nonlinear temperature profile in the slice. For a heat content of 2 x 10^4 cal cm^{-2}, which is a reasonable summertime value, the positive error of 140 cal cm^{-2} in the 7-8 meter layer is 0.7% of the heat content. Assume that the mean temperatures calculated at each level are 0.2°C higher than the true means due to sampling bias. This represents a positive error of 250 cal cm^{-2}, or 1.2% of the heat content. If the individual interpolation errors cancel in the remaining layers, then the computed value of θ is about two percent higher than the true heat content. This hypothetical case is rather extreme, but the result indicates that differences between Q (or Q') and θ of less than two percent cannot be regarded as significant.

It is apparent from Table 6(b) that a large majority of the two-layer heat content values are reasonably close to θ, since 32 of the 46 values are within two percent of θ. All but one of the corrected values Q' fall within two percent of θ. The correction method appears to be very effective in reducing large differences between Q and θ, as a comparison of Q and Q' for four of the Tub Lake soundings reveals. Negative differences of 5.9% to 12.2% are replaced by differences of -0.1% to +1.3%. There is a slight overcorrection in three cases, but the positive differences are too small to be significant. In thirteen of the soundings the correction method actually increases the percentage difference, but the increase is insignificant in all cases since neither difference exceeds two percent in any instance.

The contributions of hypsometric curve-fitting errors to the differences between Q and Θ were investigated by calculating the Bryson-Dutton heat content from the actual soundings but replacing the weighting factors w_i by h_i^p. This heat content value is denoted by Θ_p to distinguish it from Θ. Mathematically

$$\Theta_p = H\int_0^1 c h^p T(h)\, dh \qquad (14)$$

Table 7 shows the values of Θ_p and the percentage differences between Θ_p and Θ for 40 of the 46 soundings. Rocky Lake was not included since it is virtually isothermal throughout the year, and Clear Lake was omitted since the curve fit is nearly perfect in the most heavily weighted layers. The differences vary from -0.2% to +1.6%, not significantly greater than the uncertainty of Θ itself, and in the Tub Lake soundings for which Q has been found to be more than five percent smaller than Θ, almost all of the

TABLE 7

VALUES OF Θ_p AND DIFFERENCES RELATIVE TO Θ

Sounding		Θ_p (10^4 cal cm^{-2})	Diff. (%)
Tub Lake	7/12/58	.607	+0.6
	8/ 7/58	.698	+0.5
	9/ 3/58	.560	+0.1
	6/12/59	.566	+0.7
	7/17/59	.663	+0.3
	8/18/59	.642	0.0
	6/14/60	.489	+0.9
	9/ 7/60	.598	+0.3
Zed Lake	7/17/58	1.109	+1.3
	8/ 2/59	1.261	+1.1
Waskesiu Lake	7/21/59	1.880	-0.1
	5/28/60	.753	-0.1
	6/22/60	1.304	-0.1
	7/ 9/60	1.673	-0.1
	7/18/60	1.944	-0.2
	8/ 9/60	2.030	0.0
Lake Mendota	6/30/97	2.217	+1.2
	8/ 5/98	2.309	+1.3
	7/ 9/06	2.157	+1.1
	7/30/06	2.520	+1.2
	7/ 5/11	2.478	+1.5
	6/26/12	2.153	+0.8
	7/31/12	2.404	+1.0
	7/18/14	2.435	+1.4
	9/ 7/14	2.314	+0.7
Grace Lake	8/19/59	.687	+1.6
	7/27/60	.831	+1.6
	6/27/61	.830	+1.5
Trout Lake	5/30/58	1.341	+0.3
	5/19/58	1.164	+0.1
	7/ 3/59	2.037	+0.4
	8/25/59	2.432	+0.4
	7/ 5/60	1.762	+0.4
	8/17/61	2.152	+0.4
Deer Lake	9/ 9/58	2.144	+0.1
	5/ 4/59	.814	+0.3
	7/20/59	2.378	+0.6
	8/24/59	2.428	+0.5
	9/26/59	1.985	+0.1
	8/19/60	2.205	+0.8

discrepancy results from replacing T by T'. As would be expected from the vertical distributions of curve-fitting error, the largest positive differences for θ_p occur for Zed Lake, Lake Mendota and Grace Lake, in which lakes w is overestimated in the upper halves and the positive errors of w' are accentuated by the higher values of T multiplying w' as h increases. For Tub Lake and Deer Lake, the differences are positive but smaller since the effects of the negative curve-fitting errors for $.5 < h < .8$ partially neutralize the effects of positive ones for higher values of h. For Trout Lake and Waskesiu Lake, θ_p and θ are within 0.5% of each other.

A sounding for which the two-layer method yields good results is the Lake Mendota sounding of July 9, 1906. As listed in Table 6(b), both Q and Q' are well within two percent of θ. The difference between T' and T reaches 4°C just above the level $h = h^*$ and -4°C just below this level, but the sounding shows strong overall cancellation of temperature errors. The shape of the sounding closely resembles that of the schematic summer sounding of Figure 1, and the corresponding residual regions are similar in shape. The correction scheme, which is unnecessary in this case, practically eliminates what little discrepancy there is between Q and θ, merely replacing one good estimate by another good one. The positive errors of w' in the upper layers make Q

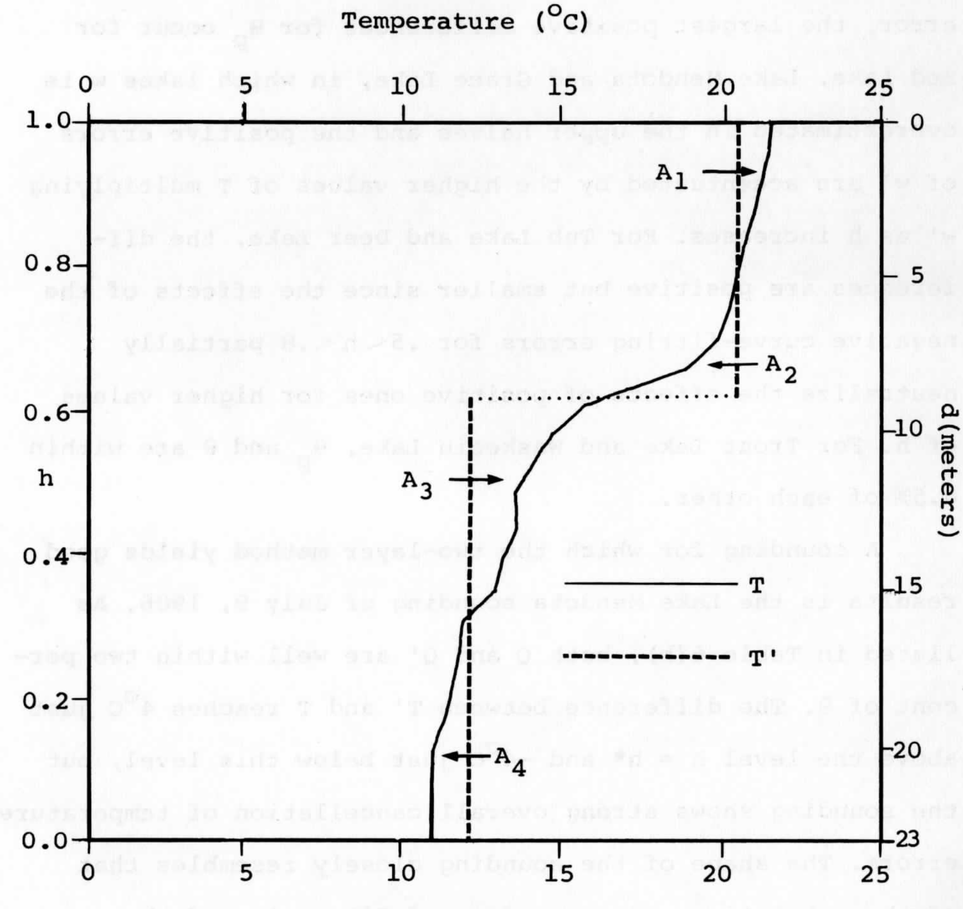

Figure 6. Lake Mendota sounding of July 9, 1906.

and Q' only 1.1% greater than they would be if the hypsometric curve fit were perfect. Neither approximation by itself tends to produce any difference between Q and θ that is materially larger than the uncertainty of θ.

Figure 7 displays the Tub Lake sounding of June 14, 1960, in which Q is 12.2% smaller than θ. This is the largest percentage difference encountered among the soundings. The temperature range of 21.1°C from surface to bottom is extremely large, but the thermocline is diffuse rather than concentrated and begins at the surface. Since θ_p is within one percent of θ, the large negative difference is completely due to the nature of the two-layer temperature curve. The extensive and large underestimate of temperature indicated by the size of A_3 is outstanding. T' averages about 3°C too low in the corresponding interval, and the pointwise error of T' reaches -10°C just beneath the level h*. Despite the shallowness of the thermocline, the shape of the sounding is similar to that of the schematic sounding in Figure 1, except in the short interval corresponding to A_2, and the correction method is highly effective.

The Deer Lake sounding of May 4, 1959, shown in Figure 8, has a very small temperature spread, from 7.4°C at the surface to 5.1°C at the bottom, and cannot be said to possess a thermocline, but Q is more than three percent smaller than θ. The reason for the pronounced negative difference is much the same as in Figure 7, with an

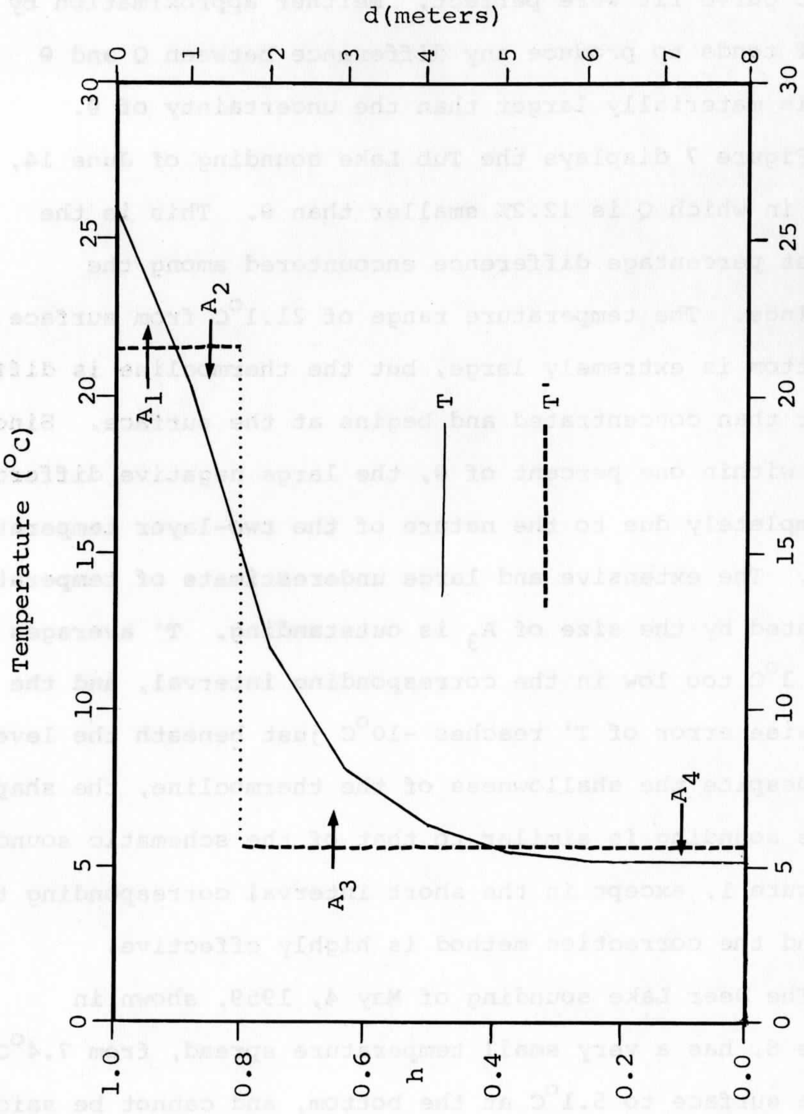

Figure 7. Tub Lake sounding of June 14, 1960.

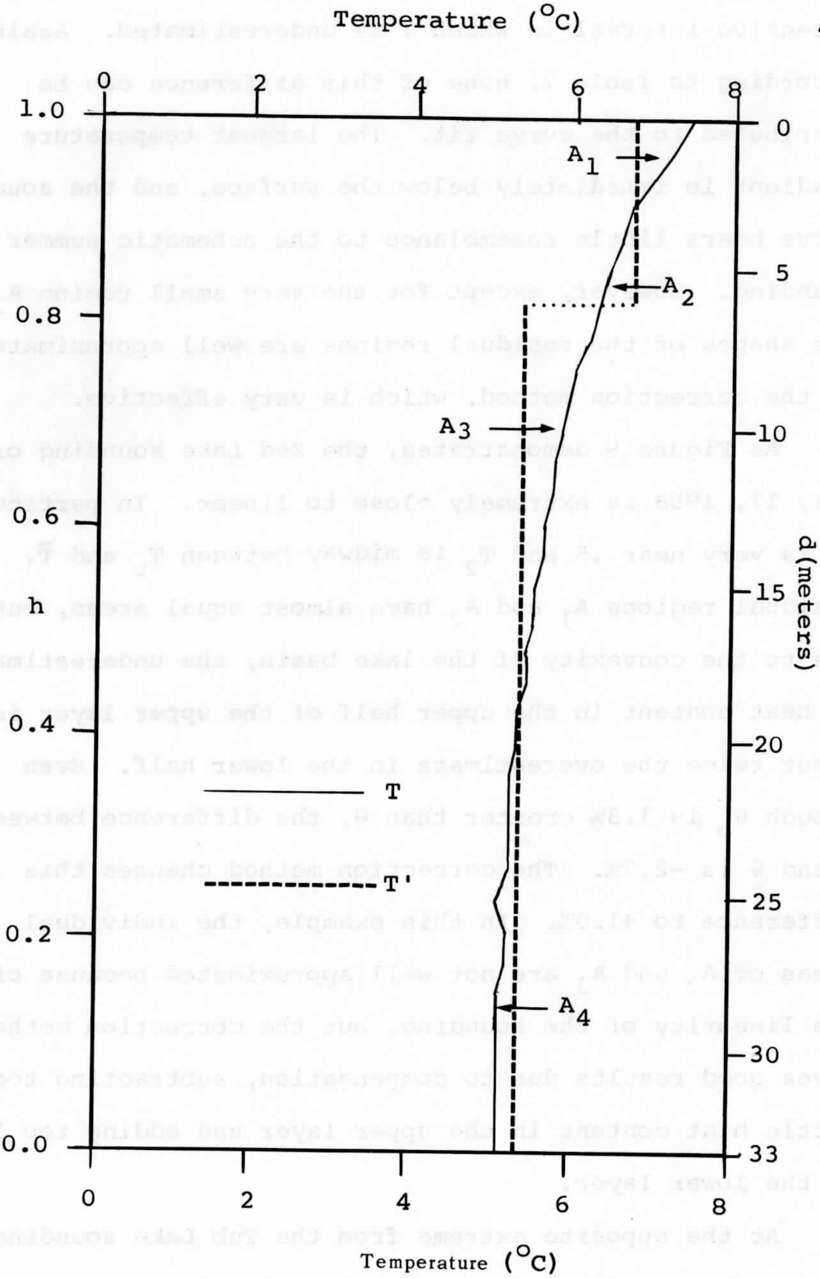

Figure 8. Deer Lake sounding of May 4, 1959.

extensive interval in which T is underestimated. Again, according to Table 7, none of this difference can be attributed to the curve fit. The largest temperature gradient is immediately below the surface, and the sounding curve bears little resemblance to the schematic summer sounding. However, except for the very small region A_2, the shapes of the residual regions are well approximated in the correction method, which is very effective.

As Figure 9 demonstrates, the Zed Lake sounding of July 17, 1958 is extremely close to linear. In particular, h* is very near .5 and T_2 is midway between T_t and $\bar{T}$. The residual regions A_1 and A_2 have almost equal areas, but due to the convexity of the lake basin, the underestimate of heat content in the upper half of the upper layer is about twice the overestimate in the lower half. Even though θ_p is 1.3% greater than θ, the difference between Q and θ is -2.7%. The correction method changes this difference to +1.0%. In this example, the individual areas of A_2 and A_3 are not well approximated because of the linearity of the sounding, but the correction method gives good results due to compensation, subtracting too little heat content in the upper layer and adding too little in the lower layer.

At the opposite extreme from the Tub Lake sounding, the Grace Lake sounding of July 27, 1960 in Figure 10 shows the

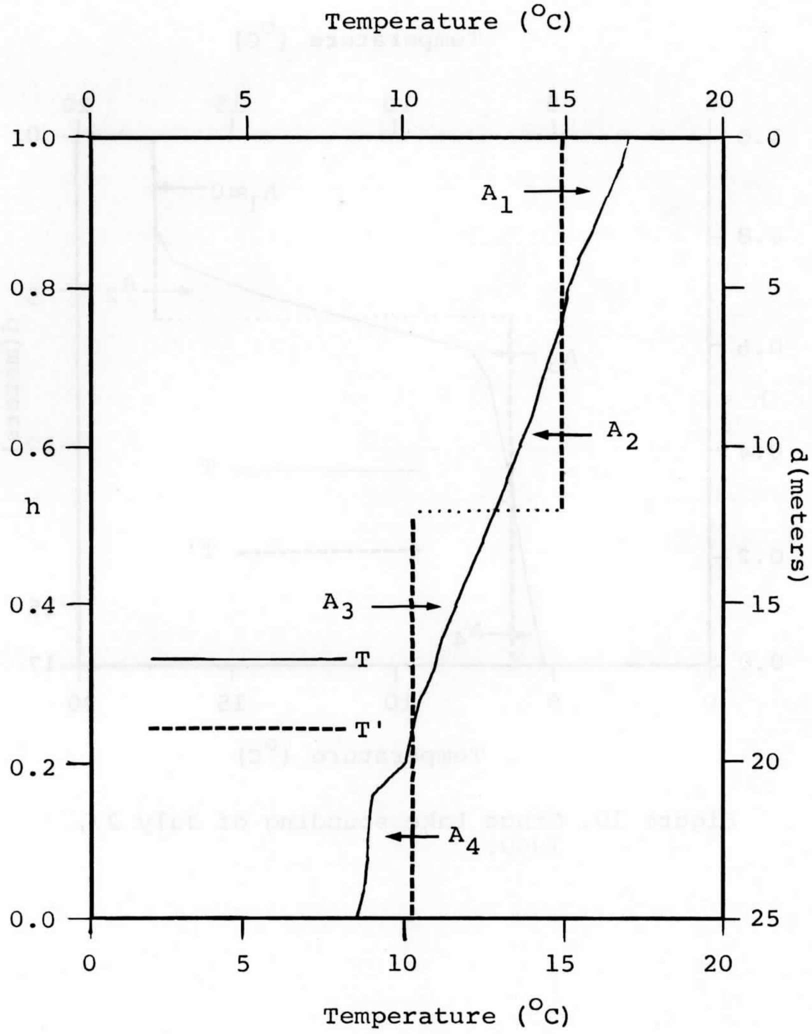

Figure 9. Zed Lake sounding of July 17, 1958.

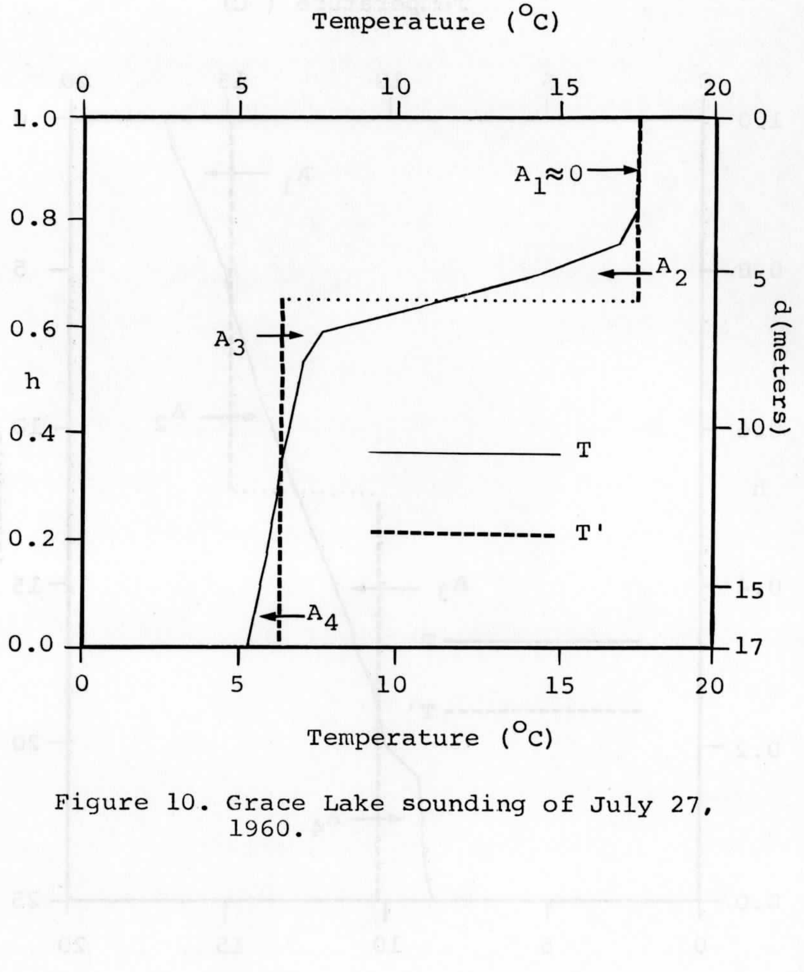

Figure 10. Grace Lake sounding of July 27, 1960.

greatest positive difference (3.4%) between Q and θ. In this case, the overestimation of w by the power curve through the upper half of the basin contributes considerably to the difference. Otherwise, Q is too large because of the temperature approximation, which overestimates T by an average of about 2.5°C in the lower half of the upper layer. The correction method is seen from Table 6(b) to be relatively ineffective. Although the sounding in Figure 10 resembles the schematic curve in Figure 1, too little heat content is subtracted in the upper layer and too much is added in the lower layer, so that there is aggravation rather than compensation of two error contributions. Partly due to this aggravation and partly due to positive error of curve fit, little of the total heat content difference is removed.

6. Discussion

In this section, the practical advantages and potential uses of the two-layer method will be considered.

The two-layer method, as presented earlier in this paper, has assumed that a complete sounding of a lake is available. However, a lake which has a thermocline can be partitioned in much the same way if the only known data are the hypsometric measurements, surface temperature, bottom temperature and thermocline depth. Then, in equation (12), T_b and T_t replace T_1 and T_2 respectively, while the thermocline depth plays the role of h*. The surface temperature can be measured conveniently from a boat at various points on the lake. Thermocline depth can be estimated from the fetch, or partial soundings can be taken at several locations once or twice a month to obtain actual thermocline depths. Below the thermocline, temperature changes are small both in space and in time. Therefore, monthly measurements below the thermocline depth are usually sufficient to reveal the gross variation of temperature at or near the deepest point during the thermocline season. Values between observation dates can be interpolated.

The method is easily programmed on FORTRAN cards. In view of its mathematical simplicity, no special subprograms are needed. PROGRAM MINIMAL, a program shown in the Appendix, makes use of only the minimal data referred to in the preceding paragraph. The program was run on the IBM 1604 computer at the University of Wisconsin Computing Center. About three minimum-data soundings per second can be processed, suggesting the close to a thousand heat content computations can be performed in five minutes. The number of calculations needed to compute the heat content is independent of the number of hypsometric data points, since the hypsometric data are used only once to compute the normalized volume of a lake. Although the Bryson-Dutton method is also easily programmed, each heat content calculation uses the hypsometric data and requires as many multiplications as there are data points. If there are many data points, computer time can be reduced by using the two-layer method.

Minimal data from 35 of the 46 soundings referred to earlier were used to obtain heat content values. The surface temperature T_t was frequently appreciably higher than T_2 and therefore less representative of the upper layer, but the results were reasonable. The heat content values were within two percent of the Bryson-Dutton values in 21 cases, with differences of more than four percent in only four instances. The results also suggest that in cases of large temperature gradients near the surface, replacing T_2 by T_t has the desirable effect of replacing large negative differences by considerably smaller positive differences. It appears that much of the data gathering involved in obtaining frequent complete soundings of lakes can be eliminated with no serious loss of accuracy in heat content calculations.

The two-layer method can be applied to heat balance considerations. According to Bryson and Dutton (1960), if heat flux by chemical and biological processes and by horizontal advection is ignored, then

$$E = (R - C)/(1 + B) \qquad (15)$$

where R is net radiation, C is the time derivative of heat content, B is the Bowen ratio and E is evaporation. If all quantities on the right hand side of equation (15) are known, then E can be computed easily without having to evaluate eddy diffusivities. The average value of C between two dates is readily calculated from the heat content values for the two dates. Uncertainties in the heat content values limit the accuracy of the computed value of C, but judicious smoothing of heat content curves obtained by the two-layer method can alleviate the difficulty.

The discharge of heated water from nuclear power plants poses a serious threat to the ecological balance of lakes. The lake water can become too warm to support cold-water fish but more favorable for the growth of undesirable organisms. One way of gauging the degree of thermal pollution over several years or longer would be to gather thermal data during successive summers to check for possible systematic upward trends in the maximum heat content. If there is no corresponding climatic warming trend, a year-to-year increase in heat content would be a measure of thermal pollution.

The three critical dates in the annual cycle of a temperate lake are the dates of break-up, maximum heat content and freeze-up. The findings of McFadden (1965) for deep Canadian lakes indicate that break-up occurs within a few days of the date when the forty-day running mean air temperature rises to 4.5°C, and freeze-up occurs within one to three days of the date when the forty-day running mean falls to 0°C. It is not yet known if a similar type of correlation exists between the date of maximum heat content and meteorological or geographical variables. Assume that surface temperatures are measured every two or three days during the thermocline season, and that thermocline depths and hypolimnion temperatures are checked regularly about every three weeks. Then the two-layer method will yield enough heat content values to pinpoint maximum dates more closely than has been possible on the basis of complete soundings taken about once monthly. Given several years of data for a variety of lakes at different latitudes, possible correlations between the date of maximum heat content and factors such as running mean air temperature, wind speed or latitude can be investigated.

7. Conclusions

On the basis of the results tabulated and described earlier, the following conclusions regarding the two-layer method may be stated.

1) Assuming complete temperature soundings are used as described in the paper:

a) The method usually gives satisfactory estimates of heat content without the correction scheme. The difference between the Bryson-Dutton heat content and the two-layer heat content is usually not significantly larger than the uncertainty of the Bryson-Dutton value, chiefly due to partial cancellation of temperature differences. Substantially larger negative differences can result when the temperature gradient is largest just below the surface.

b) For a variety of basin shapes, the effect of the hypsometric curve-fitting errors on the computed heat content is masked by the uncertainty of the Bryson-Dutton value. Large differences between the values are therefore due almost entirely to errors of temperature approximation.

c) The correction scheme effectively eliminates significant differences even when it does not closely estimate the areas of the individual residual regions.

2) The method is readily programmed on FORTRAN cards, requiring no special subprograms. Since the hypsometric data are used only to compute the volume of a lake, the two-layer method generally requires less computation than does the Bryson-Dutton method, especially if many hypsometric data points are used.

3) With slight modification, the two-layer method

can be applied to a lake having a thermocline given only the hypsometric data, surface and bottom temperatures and the thermocline depth. Much of the data gathering necessary for frequent complete soundings can be avoided with no serious loss of accuracy.

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8. Appendix. FORTRAN program for modified two-layer method using minimal data

```
      PROGRAM MINIMAL
      DIMENSION DEPTH(36),W(36),WBAR(36),DELH(36),SLICE(36),XSTA(5)
C     THE NUMBER IN PARENTHESES FOLLOWING EACH VARIABLE NAME EXCEPT
C     **XSTA** MUST BE THE SMALLEST MULTIPLE OF 12 NOT EXCEEDING
C     THE LARGEST NUMBER OF HYPSOMETRIC DATA POINTS FOR ANY OF
C     THE LAKES USED IN ANY ONE PROGRAM. THE NUMBER
C     IN PARENTHESES AFTER **XSTA** MUST ALWAYS BE 5
C     THIS PROGRAM APPROXIMATES THE HEAT CONTENT OF A LAKE HAVING A
C     THERMOCLINE, ASSUMING THE ONLY KNOWN DATA ARE SURFACE AND BOTTOM
C     TEMPERATURES, THERMOCLINE DEPTH AND HYPSOMETRIC DATA
C     ALL FORMAT SPECIFICATIONS IN THIS PROGRAM REFER TO THE **PRINT**
C     OR **READ** STATEMENT IMMEDIATELY PRECEDING
      JOT=0
   32 JOT=JOT+1
      DO 14 KN=1,36
      DEPTH(KN)=0.
      W(KN)=0.
      WBAR(KN)=0.
      DELH(KN)=0.
   14 SLICE(KN)=0.
C     READ IN THE NAME OF THE LAKE AND THE SMALLEST MULTIPLE OF 12
C     NOT EXCEEDING THE NUMBER OF HYPSOMETRIC DATA POINTS. USE A DUMMY
C     CARD WITH ZERO FOR THIS NUMBER TO STOP THE PROGRAM WHEN DESIRED
      READ 9,XSTA(I),I=1,5),NX
    9 FORMAT(5A5,I3)
C     READ IN THE DEPTHS FOR HYPSOMETRIC POINTS (METERS)
C     ROUNDED TO THE NEAREST TENTH
C     IN INCREASING ORDER, FROM SURFACE (0.0 METERS) TO DEEPEST
C     POINT. PUNCH 12 NUMBERS ON EACH CARD ACCORDING TO FORMAT. IF
C     THE GREATEST DEPTH OCCURS BEFORE THE 12TH ENTRY ON THE
C     LAST CARD FOR A GIVEN LAKE, PUNCH THE VALUE OF THE GREATEST DEPTH
C     FOR THE REMAINING ENTRIES
```

```
      IF(NX)999,999,100
  100 DO 76 JUMP=1,NX,12
      JJ=JUMP+11
   76 READ 39,(DEPTH(N),N=JUMP,JJ)
   39 FORMAT(12F6.1)
C     READ IN THE MAXIMUM DEPTH (CENTIMETERS)
      READ 2,HMAX
    2 FORMAT(F9.1)
C     READ IN THE AREA WEIGHTS FOR HYPSOMETRIC POINTS
C     USING THREE DECIMAL PLACES AFTER THE DECIMAL
C     IN AN ORDER CORRESPONDING TO THE DEPTHS (SURFACE TO BOTTOM)
C     PUNCH 12 NUMBERS ON EACH CARD. IF THE SMALLEST WEIGHT OCCURS
C     BEFORE THE 12TH ENTRY ON THE LAST CARD FOR A GIVEN LAKE,
C     PUNCH .000 FOR THE REMAINING ENTRIES.
      DO 98 NAP=1,NX,12
      NN=NAP+11
   98 READ 58,(W(I),I=NAP,NN)
   58 FORMAT(12F6.3)
      NO=NX-1
C     COMPUTE NORMALIZED VOLUME OF LAKE AND CORRESPONDING EXPONENT
      DO 285 J=1,NO
      WBAR(J)=0.5*(W(J)+W(J+1))
      DELH(J)=100.*(DEPTH(J+1)-DEPTH(J))/HMAX
  285 SLICE(J)=WBAR(J)*DELH(J)
      MOE=0
  150 MOE=MOE+1
      IF(MOE-1)40,40,13
   40 V=SLICE(MOE)
      GO TO 150
   13 V=V+SLICE(MOE)
      IF(MOE-NO)150,321,321
  321 P=(1./V)-1.
```

```
   11 PRINT 178,(XSTA(I),I=1,5)
  178 FORMAT(3X,5A5)
C     READ IN GEOGRAPHIC CO-ORDINATES OF LAKE
C     LATITUDE AND LONGITUDE IN DEGREES TO THE NEAREST TENTH
      READ 51,LAT,LONG
   51 FORMAT(2F7.1)
      PRINT 20,P,DEPTH(NX),LAT,LONG
   20 FORMAT(3X,2HP=,F5.3,3X,9HMAX DEPTH,F6.1,1X,1HM,
     16X,3HLAT,F5.1,4X,4HLONG,F6.1)
      PRINT 6
    6 FORMAT(3X,8HMO DA YR,4X,3HSFC,5X,3HBOT,5X,6HTCLINE,5X,2HH*,5X,
     112HHEAT CONTENT)
      PRINT 82
   82 FORMAT(15X,4HTEMP,4X,4HTEMP,5X,5HDEPTH,13X,9H(2 LAYER))
      PRINT 126
  126 FORMAT(15X,4HDEGC,4X,4HDEGC,7X,1HM,15X,9HCAL/SQ CM)
      IP=0
   70 IP=IP+1
C     READ IN OBSERVATION DATE, SURFACE AND BOTTOM TEMPERATURES AND
C     THERMOCLINE DEPTH
C     TEMPERATURES IN DEGREES C, THERMOCLINE DEPTH IN METERS
C     DATE FORMAT -- NUMBER OF MONTH, NUMBER OF DAY, LAST TWO
C     FIGURES OF THE YEAR. OTHER DATA ROUNDED TO NEAREST TENTH
      READ 404,MO,NDA,NYR,T2,T1,DCLIN
  404 FORMAT(3I4,3F6.1)
C     COMPUTE TWO-LAYER HEAT CONTENT IN CAL PER SQ CM
      IF(T2-99.)110,32,32
  110 ZCLIN=HMAX-100.*DCLIN
      HCLIN=ZCLIN/HMAX
      R=T2-T1
      Q=(HMAX*T2-R*ZCLIN*(HCLIN**P))/(P+1.)
```

```
C     THE QUANTITY Q IS, FOR PRACTICAL PURPOSES, NUMERICALLY EQUAL TO
C     THE HEAT CONTENT IN THE C-G-S SYSTEM, EVEN THOUGH ITS DIMENSIONS
C     DIFFER. BOTH FACTORS OF THE OMITTED MULTIPLIER (DENSITY TIMES
C     SPECIFIC HEAT CAPACITY FOR FRESH WATER) ARE VERY NEARLY 1.00
      PRINT 28,MO,NDA,NYR,T2,T1,DCLIN,HCLIN,Q
   28 FORMAT(3X,I2,1H/,I2,1H/,I2,2F8.1,F10.1,F8.3,5X,E9.2)
C     THE NEXT DATA CARD FOR THE PRESENT LAKE WILL BE READ IN, UNLESS
C     A DUMMY DATA CARD, WITH A FICTITIOUS SURFACE TEMPERATURE GREATER
C     THAN 99.0C COMES UP, IN WHICH CASE THE COMPUTER WILL BE DIRECTED
C     TO THE NEXT LAKE (OR TO THE STOP CARD AT THE END OF THE DECK)
C     FOR EACH LAKE, THE DATA CARDS MUST BE ARRANGED AS FOLLOWS///
C     **1** NAME OF LAKE AND SMALLEST MULTIPLE OF 12 NOT EXCEEDING
C     THE NUMBER OF HYPSOMETRIC DATA POINTS
C     **2** SUCCESSIVE DEPTHS (METERS)
C     **3** MAXIMUM DEPTH (CENTIMETERS)
C     **4** SUCCESSIVE HYPSOMETRIC WEIGHT FACTORS CORRESPONDING
C     TO DEPTHS
C     **5** GEOGRAPHIC DATA
C     **6** DATE, SURFACE TEMPERATURE, BOTTOM TEMPERATURE AND
C     THERMOCLINE DEPTH. ONE CARD FOR EACH DATE
C     **7** DUMMY CARD, WITH A FICTITIOUS SURFACE TEMPERATURE GREATER
C     THAN 99.0C TO DIRECT COMPUTER TO NEXT LAKE
C     AFTER THE DUMMY CARD FOR THE LAST LAKE, A STOP CARD MUST APPEAR,
C     BEARING A ZERO IN COLUMN 28 AND ANY DUMMY LAKE NAME DESIRED
C     (CHOICE IMMATERIAL) ENDING IN COLUMN 25
      GO TO 70
  999 STOP
      END
```

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13. ABSTRACT Given a non-isothermal sounding, a lake is partitioned systematically into layers. The temperatures at their vertical midpoints are taken as representative for each layer. The hypsometric data are fitted to a simple power profile such that the actual and approximating basins have equal volumes. By the use of these simple functions, the integral which defines heat content is easily evaluated in cosed form. (U)

The two-layer method is tested on 46 open-season soundings for nine temperate North American lakes. The two-layer heat content values are compared to those obtained by the method of Bryson & Dutton (1960) in their analysis of Lake Mendota. The results suggest that the two-layer heat content values are not significantly different from the Bryson-Dutton values except for soundings having the steepest temperature gradient near the surface. An effective scheme is devised to correct for large differences. The shapes of the areas bounded by a schematic summer sounding and the two-layer curve are considered. (U)

The method lends itself readily to FORTRAN computer programming, and no special subprograms are needed. Although the method as presented requires complete soundings, it can be applied with slight changes to lakes with thermoclines when only the hypsometric data, surface temperature, bottom temperature & thermocline depth are known

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14. KEY WORDS	LINK A		LINK B		LINK C	
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