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FROM: Thomas O. Haig
Executive Director

REFERENCE: Contract NAS5-21798

SUBJECT: Monthly Progress Report for "Studies of Soundings and Imaging Measurements from Geostationary Satellites"

Task A <u>Investigation of Meteorological Data Processing Techniques</u>

The radiance calculation program has been modified for efficient use in developing multi-channel cloud height determination techniques. Minor bugs have been eliminated and work is proceeding on optimization of channels, effects of noise, and sensitivity to temperature profile uncertainties.

Preliminary tests of an earth edge detection algorithm have been made in an attempt to correct ATS line start jitter. The technique uses a threshold detection scheme which is relatively insensitive to line to line variations of noise in the data. A fixed threshold is added to the mean noise present in the space view. The earth edge location is then determined by locating the first in a sequence of five samples in a row which all exceed the threshold. We are currently varying the threshold to determine the optimal range which is not significantly affected by either noise on the low side or cloud brightness variations on the high side.

Task B <u>Sun Glitter</u>

Work is proceeding on geometrical aspects of the problem. Complete results are not yet available.

Correction should be made to monthly report for October which indicated that ATS-III images were being used in this study. Instead, this study is concerned with developing techniques for analysis of sun glitter information frow low orbit satellites. Data being used are from NIMBUS and ITOS.

Task D <u>Cloud Growth Rate</u>

An analysis on the distribution of ATS brightness elements as a function of time in a mid-latitude storm complex has since been completed. With McIDAS presently in commission, we hope to make rapid progress in this task next month.

Task E <u>Comparative Studies in Satellite Stability</u>

Part IV of the results of this study is attached. This paper accurately reports the portion of the work performed by Mr. Das during the first quarter of the current contract. It was noted in the quarterly report submitted last month that this segment of Mr. Das' study would be included with this monthly report since it was not possible to complete necessary reviews of this work in time for the quarterly report. Three more segments of Mr. Das' study remain to be completed, and he is proceeding on this work.

Task G <u>Rainfall Measurements by RAKE Radar</u>

As noted in earlier reports, work on this task has been terminated.

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Enclosure: Part IV, Motion Analysis and Control

PART IV

MOTION ANALYSIS AND CONTROL

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Nomenclature

Notation not defined here is defined in the previous reports.

$[I]$	=	Identity matrix
$[\delta_B^*]$	=	Matrix for the body B, defined by Eq. (4.3)
$[\delta_{Bi}]$, $(i = 1\text{-}6)$	=	Matrices for the body B, defined by Eqs. (4.4) and (4.5)
$\underline{\theta}$	=	Angular displacements of the body A relative to the body B
$[\delta_A^*]$	=	Matrix for the body A, defined by Eq. (4.8)
γ_1	=	$\dfrac{\lVert M\rVert_B}{\lVert M\rVert_A}$
$[\delta_{Ai}]$, $(i = 1\text{-}6)$	=	Matrices for the body A, defined by Eqs. (4.16) and (4.17)
$[S_{Ai}]$, $(i = 1\text{-}5)$; $\underline{u}_A$	=	Matrices and vector for the body A, defined in Eq. (4.22)
$[S_{Bi}]$, $(i = 1\text{-}5)$; $\underline{u}_B$	=	Matrices and vector for the body B, defined in Eq. (4.24)
$\underline{\theta}^*$	=	Initial value of $\underline{\theta}$
$g(t)$	=	Vector valued function, defined in Eq. (4.26)
$F_1(t)$, $F_2(t)$	=	Vector valued functions, defined in Eq. (4.32)
$\underline{x}_i$, $i = 0,1,2,\ldots$	=	Components of $\underline{\omega}_B$, defined in Eq. (4.42)
$\underline{u}_{Ai}$, $i = 0,1,2,\ldots$	=	Components of $\underline{u}_A$, defined in Eq. (4.43)
$\underline{u}_{Bi}$, $i = 0,1,2,\ldots$	=	Components of $\underline{u}_B$, defined in Eq. (4.44)
ε'	=	Maximum value of the external torques

$\underline{f}(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{\theta}, \underline{u}_A, \underline{u}_N)$	=	Nonlinear terms of Eq. (4.35), defined in Eq. (4.46)
$[B_i]$, $i = 1\text{-}7$	=	Matrices, defined in Eq. (4.46a)
$\underline{v}_i$, $i = 0,1,2,\ldots$	=	Vectors, defined in Eq. (4.48)
$\underline{w}_i$, $i = 0,1,2,\ldots$	=	Vectors, defined in Eq. (4.53)
$\phi_1(t)$, $\phi_2(t)$	=	Fundamental matrices, defined in Eq. (4.54)
M_i, $i = 0,1,2,\ldots$	=	$\lVert\underline{w}_i\rVert$
$\underline{x}_0^*(t)$	=	Vector valued function, defined in Eq. (4.60)
$\psi(t)$	=	$[\phi_2(0), \dot{\phi}_2(t)]^T$
$\underline{\mu}_1$	=	Vector valued multiplier, defined in Eq. (4.65)

1. Introduction

In our previous report the long-time solution series for $\underline{\omega}$ was assumed as

$$\underline{\omega} = \underline{\omega}_0 + \varepsilon\underline{\omega}_1 + \varepsilon^2\underline{\omega}_2 + \ldots \tag{3.11}$$

where $\underline{\omega}_0$ was found to be given by the equation

$$\underline{y}(\underline{\omega}_0, t) = 0 \tag{3.20}$$

and $\underline{\omega}_1$, $\underline{\omega}_2$, etc. were obtained as functions of $\underline{\omega}_0$. The vector $\underline{y}$ is a function of the controlling torques also. In the following analysis the explicit expressions for $\underline{y}$ for each of the bodies A and B will first be obtained. Then with a suitable control policy, the control torques will be optimized and expressed as functions of $\underline{\omega}_0$.

2. The Form of $\underline{y}(\underline{\omega}_0, t)$ for the Body B.

For the body B, $\underline{y}(\underline{\omega}_0, t)$ is the vector $\underline{M}_{B3}(\underline{\omega}_B, t)$. The Eqs. (2.115) and (2.115a) gives $\underline{M}_{B3}$ as

$$\underline{M}_{B3} = A_{B4} - A_{B1}(A_{B1} + P_{B1})^{-1}(A_{B4} + P_{B4}) \tag{4.1}$$

In the above equation, the values of A_{B1} and A_{B4} are substituted from Eq. (2.111a), so that

$$M_{B3} = L_B^T\bar{A}_4 - (L_B^T\bar{A}_1L_B)[L_B^T\bar{A}_1L_B + P_{B1}]^{-1}[L_B^T\bar{A}_4 + P_{B4}] \tag{4.2}$$

The elements of L_B, $\bar{A}_1$ and P_{B1} are constants. Let $[\delta_B^*]$ be defined by

$$[\delta_B^*] = (L_B^T\bar{A}_1L_B)[L_B^T\bar{A}_1L_B + P_{B1}]^{-1} \tag{4.3}$$

From Part I, the vector $\bar{A}_4$, a term of Eq. (1.116), can be expressed in the form

$$\underline{\bar{A}}_4 = [\delta_{B1}]\dot{\underline{\omega}}_B + \tilde{\omega}_B[\delta_{B2}]\underline{\omega}_B + [\delta_{B3}]\dot{\underline{\theta}} + [\delta_{B4}]\underline{\theta} \tag{4.4}$$

The vector $\underline{P}_{B4}$ is given by Eq. (2.104) in the form

$$\underline{P}_{B4} = (\underline{T}_B^* - \dot{\underline{h}}_B) - [I_B]\dot{\underline{\omega}}_B + [\tilde{h}_B]\underline{\omega}_B - \tilde{\omega}_B[I_B]\omega_B - [\delta_{B5}]\dot{\underline{\theta}} - [\delta_{B6}]\underline{\theta} + \underline{T}_{EB} \quad (4.5)$$

In these equations, $\underline{\omega}_B$ stands for the angular velocity components of $\underline{\omega}_0$ of the body B. Substituting Eqs. (4.3), (4.4) and (4.5) in Eq. (4.2), we get

$$\underline{M}_{B3} = (I - \delta_B^*)L_B^T\{[\delta_{B1}]\dot{\underline{\omega}}_B + \tilde{\omega}_B[\delta_{B2}]\underline{\omega}_B + [\delta_{B3}]\dot{\theta} + [\delta_{B4}]\underline{\theta}\}$$

$$- [\delta_B^*]\{(\underline{T}_B^* - \dot{\underline{h}}_B) - [I_B]\dot{\underline{\omega}}_B + [\tilde{h}_B]\underline{\omega}_B - \tilde{\omega}_B[I_B]\underline{\omega}_B - [\delta_{B5}]\dot{\underline{\theta}} - [\delta_{B6}]\underline{\theta} + \underline{T}_{EB}\}.$$

Therefore, as

$$\underline{M}_{B3} = \underline{0}$$

we have

$$\underline{0} = \{(I - \delta_B^*)[L_B^T][\delta_{B1}] + [\delta_B^*][I_B]\}\dot{\underline{\omega}}_B - [\delta_B^*]\tilde{h}_B\underline{\omega}_B$$

$$+ \{(I - \delta_B^*)[L_B^T][\delta_{B3}] + [\delta_B^*][\delta_{B5}]\}\dot{\underline{\theta}}$$

$$+ \{(I - \delta_B^*)[L_B^T][\delta_{B4}] + [\delta_B^*][\delta_{B6}]\}\underline{\theta} - [\delta_B^*](\underline{T}_B^* - \dot{\underline{h}}_B)$$

$$+ (I - \delta_B^*)[L_B^T]\tilde{\omega}_B[\delta_{B2}]\underline{\omega}_B + [\delta_B^*]\tilde{\omega}_B[I_B]\underline{\omega}_B - [\delta_B^*]\underline{T}_{EB} . \quad (4.6)$$

In Eq. (4.6), $\underline{T}_{EB}$ is a highly nonlinear vector. This vector equation has only three nonzero scalar equations. This can readily be seen from Eq. (4.1), which is

$$\underline{M}_{B3} = \underline{A}_{B4} - [(A_{B1} + P_{B1})A_{B1}^{-1}]^{-1}(\underline{A}_{B4} + \underline{P}_{B4})$$

$$= \underline{A}_{B4} - [I + P_{B1}A_{B1}^{-1}]^{-1}(\underline{A}_{B4} + \underline{P}_{B4})$$

$$= \underline{A}_{B4} - [I - P_{B1}A_{B1}^{-1} + (P_{B1}A_{B1}^{-1})^2 - \ldots](\underline{A}_{B4} + \underline{P}_{B4})$$

or

$$\underline{M}_{B3} = - \underline{P}_{B4} + P_{B1}A_{B1}^{-1}(I + P_{B1}A_{B1}^{-1})(\underline{A}_{B4} + P_{B4}) \quad (4.7)$$

As $\underline{P}_{B4}$ is a (3×1) vector and $[P_{B1}]$ is a $(3 \times n)$ matrix, so $\underline{M}_{B3}$ is also a (3×1) vector.

3. The Form of $\underline{y}(\underline{\omega}_0,t)$ for the Body A

For the body A, $\underline{y}(\omega_0,t)$ is the vector $\underline{M}_{A3}(\underline{\omega}_A,t)$. Let $[\delta_A^*]$ be defined by

$$[\delta_A^*] = (L_A^T \bar{A}_1' L_A)[L_A^T \bar{A}_1' L_A + P_{A1}]^{-1} \tag{4.8}$$

Then

$$\underline{M}_{A3} = L_A^T \underline{\bar{A}}_4' - [\delta_A^*][L_A^T \underline{\bar{A}}_4' + \underline{P}_{A4}] \tag{4.9}$$

It has been found that $\underline{P}_{A4}$ and $\underline{\bar{A}}_4'$ involves the contact torques $\underline{T}_{AC}$, and the contact forces $\underline{F}_{AC}$. The expressions for these given by Eqs. (1.104) and (1.106) involve the transformation matrices Θ_{AB} and Θ_{AB}^{-1} which are nonlinear functions of the angular velocities. To keep Eq. (4.9) free of Θ_{AB} and Θ_{AB}^{-1}, the vectors $\underline{T}_{AC}$ and $\underline{F}_{AC}$ are now reformulated directly in the A-based coordinates.

The stiffness matrices τ_i, i = 1-12, introduced in Eq. (1.96), are independent of the base vectors. We also note that if $\underline{\theta}$ is a relative angular position vector of the A-based coordinates with respect to the B-based coordinates, then $-\underline{\theta}$ is the relative angular position vector of B-based coordinates with respect to the A-based coordinates. Therefore corresponding to Eq. (1.96), we have

$$\begin{Bmatrix} {}_a\underline{F}_{BC} \\ {}_a\underline{T}_{BC} \end{Bmatrix} = \begin{bmatrix} \tau_1 & \tau_2 & -\tau_3 \\ \tau_4 & \tau_5 & -\tau_6 \end{bmatrix} \begin{Bmatrix} {}_a\underline{C}_B \\ \underline{C}_A \\ \underline{\theta} \end{Bmatrix} + \begin{bmatrix} \tau_7 & \tau_8 & -\tau_9 \\ \tau_{10} & \tau_{11} & -\tau_{12} \end{bmatrix} \begin{Bmatrix} {}_a\underline{\dot{C}}_B \\ \underline{\dot{C}}_A \\ \underline{\dot{\theta}} \end{Bmatrix} \tag{4.10}$$

where ${}_a\underline{F}_{BC}$, ${}_a\underline{T}_{BC}$ and ${}_a\underline{C}_B$ are the vectors $\underline{F}_{BC}$, $\underline{T}_{BC}$ and $\underline{C}_B$, respectively, expressed in the A-based coordinates. Hence

$$\underline{F}_{AC} = - {}_a\underline{F}_{BC} = -[\tau_1({}_a\underline{C}_B) + \tau_2(\underline{C}_A) - \tau_3\underline{\theta}]$$

$$- [\tau_7({}_a\underline{C}_B) + \tau_8(\dot{\underline{C}}_4) - \tau_9\dot{\underline{\theta}}]$$

or

$$\underline{F}_{AC} = (\frac{1}{\gamma_1}\tau_1 - \tau_2)\underline{C}_A + (\frac{1}{\gamma_1}\tau_7 - \tau_8)\underline{C}_A + (\tau_3\underline{\theta} + \tau_9\dot{\underline{\theta}}) \tag{4.11}$$

where

$$\gamma_1 = \frac{||M||_B}{||M||_A} . \tag{4.12}$$

We also have

$$\underline{T}_{AC} = - {}_a\underline{T}_{BC} - ({}_a\tilde{r}_{AB})({}_a\underline{F}_{BC})$$

or

$$\underline{T}_{AC} = [\frac{1}{\gamma_1}(\tau_4 + {}_a\tilde{r}_{AB}\tau_1) - (\tau_5 + {}_a\tilde{r}_{AB}\tau_2)]\underline{C}_A$$

$$+ [\frac{1}{\gamma_1}(\tau_{10} + {}_a\tilde{r}_{AB}\tau_7) - (\tau_{11} + {}_a\tilde{r}_{AB}\tau_8)]\dot{\underline{C}}_A$$

$$+ [\tau_6 + {}_a\tilde{r}_{AB}\tau_3]\underline{\theta} + [\tau_{12} + {}_a\tilde{r}_{AB}\tau_9]\dot{\underline{\theta}} \tag{4.13}$$

In this equation,

$${}_a\underline{r}_{AB} = {}_a\underline{r}_B - \underline{r}_A \tag{4.14}$$

The vectors $\underline{r}_B$ and $\underline{r}_A$ are defined in Figure 8 of Part I. Then the equation of motion of the body A corresponding to Eq. (2.104) is given by

$$[P_{A1}]\ddot{\underline{q}}_A + \{\tilde{\omega}_A[P_{A1}] - [\frac{1}{\gamma_1}(\tau_{10} + {}_a\tilde{r}_{AB}\tau_7) - (\tau_{11} + {}_a\tilde{r}_{AB}\tau_8)$$

$$+ \tilde{R}_{AC}(\frac{1}{\gamma_1}\tau_7 - \tau_8)]G_A L_A\}\dot{\underline{q}}_A + [(\tau_5 + {}_a\tilde{r}_{AB}\tau_2) - \frac{1}{\gamma_1}(\tau_4 + {}_a\tilde{r}_{AB}\tau_1)$$

$$- \tilde{R}_{AC}(\frac{1}{\gamma_1}\tau_1 - \tau_2)]G_A L_A \underline{q}_A = \underline{T}_A^* + \underline{T}_{EA} + [\tau_6 + (\tilde{R}_{AC} + {}_a\tilde{r}_{AB})\tau_3]\underline{\theta}$$

$$+ [\tau_{12} + (\tilde{R}_{AC} + {}_a\tilde{r}_{AB})\tau_9]\dot{\underline{\theta}} - I_A\dot{\underline{\omega}}_A - \dot{h}_A + \tilde{h}_A\underline{\omega}_A - \tilde{\omega}_A I_A\omega_A \tag{4.15}$$

Equation (4.15) has the same form as Eq. (2.110) which is

$$[P_{A1}]\ddot{\underline{q}}_A + [P_{A2}]\dot{\underline{q}}_A + [P_{A3}]\underline{q}_A = \{\underline{P}_{A4}\} \tag{2.110}$$

With this formulation, we can express $\underline{\bar{A}}_4'$ and $\underline{P}_{A4}$ as

$$\underline{\bar{A}}_4' = [\delta_{A1}]\dot{\underline{\omega}}_A + \tilde{\omega}_A[\delta_{A2}]\underline{\omega}_A + [\delta_{A3}]\dot{\underline{\theta}} + [\delta_{A4}]\underline{\theta} \tag{4.16}$$

$$\underline{P}_{A4} = (\underline{T}_A^* - \dot{\underline{h}}_A) - [I_A]\dot{\underline{\omega}}_A + [\tilde{h}_A]\underline{\omega}_A - \tilde{\omega}_A[I_A]\underline{\omega}_A$$

$$- [\delta_{A5}]\dot{\underline{\theta}} - [\delta_{A6}]\underline{\theta} + \underline{T}_{EA} \tag{4.17}$$

Substituting Eqs. (4.16) and (4.17) in Eq. (4.9), the equation

$$\underline{y}(\underline{\omega}_0, t) = 0$$

for the body A becomes

$$\underline{0} = \{(I - \delta_A^*)[L_A^T][\delta_{A1}] + [\delta_A^*][I_A]\}\dot{\underline{\omega}}_A - [\delta_A^*]\tilde{h}_A\underline{\omega}_A$$

$$+ \{(I - \delta_A^*)[L_A^T][\delta_{A3}] + [\delta_A^*][\delta_{A5}]\}\dot{\underline{\theta}}$$

$$+ \{(I - \delta_A^*)[L_A^T][\delta_{A4}] + [\delta_A^*][\delta_{A6}]\}\underline{\theta} - [\delta_A^*](\underline{T}_A^* - \dot{\underline{h}}_A)$$

$$+ (I - \delta_A^*)[L_A^T]\tilde{\omega}_A[\delta_{A2}]\underline{\omega}_A + [\delta_A^*]\tilde{\omega}_A[I_A]\underline{\omega}_A - [\delta_A^*]\underline{T}_{EA} \tag{4.18}$$

Now, $\underline{\omega}_A$, $\underline{\omega}_B$ and $\underline{\theta}$ are connected by the relations

$$\underline{\omega}_A = \underline{\omega}_B + \dot{\underline{\theta}} + \tilde{\omega}_B\underline{\theta} \tag{4.19}$$

and

$$\dot{\underline{\omega}}_A = \dot{\underline{\omega}}_B + \ddot{\underline{\theta}} + 2\tilde{\omega}_B\dot{\underline{\theta}} + (\dot{\tilde{\omega}}_B + \tilde{\omega}_B\tilde{\omega}_B)\underline{\theta} \tag{4.20}$$

Substituting Eqs. (4.19) and (4.20) in Eq. (4.18), we get

$$S_{A1}\dot{\underline{\omega}}_B + S_{A1}\ddot{\theta} + \{2S_{A1}\tilde{\omega}_B + S_{A2} + S_{A3} + S_{A5}\tilde{\omega}_B\delta_{A2} - S_{A5}[\widetilde{\delta_{A2}\underline{\omega}_B}]$$

$$+ \delta_A^*\tilde{\omega}_B I_A - \delta_A^*[\widetilde{I_A\omega_B}]\}\dot{\theta} + \{S_{A1}(\dot{\tilde{\omega}}_B + \tilde{\omega}_B\tilde{\omega}_B) + S_{A2}\tilde{\omega}_B + S_{A4}$$

$$+ S_{A5}\tilde{\omega}_B\delta_{A2}\tilde{\omega}_B - S_{A5}[\widetilde{\delta_{A2}\underline{\omega}_B}]\tilde{\omega}_B + \delta_A^*\tilde{\omega}_B I_A\tilde{\omega}_B - \delta_A^*[\widetilde{I_A\underline{\omega}_B}]\tilde{\omega}_B\}\underline{\theta}$$

$$+ S_{A2}\underline{\omega}_B + [S_{A5}\tilde{\dot{\theta}}\delta_{A2}\underline{\dot{\theta}} + \delta_A^*\tilde{\dot{\theta}}I_A\underline{\dot{\theta}}] + \{S_{A5}\tilde{\dot{\theta}}\delta_{A2}\tilde{\omega}_B\underline{\theta}$$

$$+ S_{A5}[\widetilde{\underline{\omega}_B\underline{\theta}}]\delta_{A2}\underline{\dot{\theta}} + \delta_A^*\dot{\theta}I_A\tilde{\omega}_B\underline{\theta} + \delta_A^*[\widetilde{\underline{\omega}_B\underline{\theta}}]I_A\underline{\dot{\theta}}\}$$

$$+ \{S_{A5}[\widetilde{\underline{\omega}_B\underline{\theta}}]\delta_{A2}\tilde{\omega}_B\underline{\theta} + \delta_A^*[\widetilde{\underline{\omega}_B\underline{\theta}}]I_A\tilde{\omega}_B\theta\}$$

$$+ \{S_{A5}\tilde{\omega}_B\delta_{A2}\underline{\omega}_B + \delta_A^*\tilde{\omega}_B I_A\underline{\omega}_B\} - [\delta_A^*]\underline{u}_A = 0 \qquad (4.21)$$

where

$$\left.\begin{aligned}
S_{A1} &= \{(I - \delta_A^*)[L_A^T]\delta_{A1} + \delta_A^*[I_A]\} \\
S_{A2} &= -\delta_A^*\tilde{h}_A \\
S_{A3} &= \{(I - \delta_A^*)[L_A^T]\delta_{A3} + \delta_A^*\delta_{A5} \\
S_{A4} &= \{(I - \delta_A^*)[L_A^T]\delta_{A4} + \delta_A^*\delta_{A6} \\
S_{A5} &= (I - \delta_A^*)[L_A^T] \\
\underline{u}_A &= \underline{T}_A^* - \underline{\dot{h}}_A + \underline{T}_{EA}
\end{aligned}\right\} \qquad (4.22)$$

4. <u>Equations for</u> $\underline{\theta}$

Equation (4.6) is rewritten as

$$S_{B1}\underline{\dot{\omega}}_B + S_{B2}\underline{\dot{\theta}} + S_{B3}\underline{\omega}_B + S_{B4}\underline{\theta} + S_{B5}\tilde{\omega}_B[\delta_{B2}]\underline{\omega}_B$$

$$+ [\delta_B^*]\tilde{\omega}_B[I_B]\underline{\omega}_B - [\delta_B^*]\underline{u}_B = 0 \qquad (4.23)$$

where

$$\left.\begin{aligned}
S_{B1} &= (I - \delta_B^*)[L_B^T][\delta_{B1}] + [\delta_B^*][I_B] \\
S_{B2} &= (I - \delta_B^*)[L_B^T][\delta_{B3}] + [\delta_B^*][\delta_{B5}] \\
S_{B3} &= -[\delta_B^*]\tilde{h}_b
\end{aligned}\right\} \qquad (4.24)$$

$$S_{B4} = (I - \delta_B^*)[L_B^T][\delta_{B4}] + [\delta_B^*][\delta_{B6}]$$

$$S_{B5} = (I - \delta_B^*)[L_B^T]$$

$$\underline{u}_B = \underline{T}_B^* - \underline{\dot{h}}_B + \underline{T}_{EB}$$

Now $\underline{\theta}$ is formally solved from Eq. (4.23) as

$$\underline{\theta} = \exp[- S_{B2}^{-1} S_{B4}t]\{\underline{\theta}^* + \int_0^t \exp[S_{B2}^{-1}S_{B4}\tau]\underline{g}(\tau)d\tau\} \quad (4.25)$$

where $\underline{\theta}^*$ is the initial value of $\underline{\theta}$, and

$$\underline{g}(t) = \delta_B^*\underline{u}_B - S_{B1}\underline{\dot{\omega}}_B - S_{B3}\underline{\omega}_B - S_{B5}\omega_B\delta_{B2}\underline{\omega}_B - \delta_B^*\omega_B I_B\underline{\omega}_B \quad (4.26)$$

Taking zero initial value of $\underline{\theta}$, we get

$$\underline{\theta} = \exp[-S_{B2}^{-1}S_{B4}t] \int_0^t \exp[S_{B2}^{-1}S_{B4}\tau]g(\tau)d\tau \quad (4.27)$$

Differentiating Eq. (4.23), we get

$$S_{B2}\underline{\ddot{\theta}} + S_{B4}\underline{\dot{\theta}} = \underline{\dot{g}} \quad (4.28)$$

Since $\underline{\dot{\theta}}$ remains approximately constant during the motion, we can take

$$\dot{g}(t) = \text{a constant} = S_{B4}\underline{\dot{\theta}}(0) . \quad (4.29)$$

Equation (4.29) will be used only where the error will not be magnified in the remaining analysis.

Integrating Eq. (4.25) by parts, we obtain

$$\underline{\theta} = \exp - S_{B2}^{-1}S_{B4}t]\{\underline{\theta}^* + [\int\exp[S_{B2}^{-1}S_{B4}\tau]d\tau]\underline{g}(t)$$

$$- S_{B4}\underline{\dot{\theta}}(0) \int d\tau \int_0^\tau \exp[S_{B2}^{-1}S_{B4}p]dp\}_0^t \quad (4.30)$$

Now

$$S_{B4}\dot{\underline{\theta}}(0) = \begin{bmatrix} S_{B4,1,1} & S_{B4,1,2} & 0 \\ S_{B4,2,1} & S_{B4,2,2} & 0 \\ S_{B4,3,1} & S_{B4,3,2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, with the approximation of Eq. (4.29), we get

$$\underline{\theta} = F_1(t)\underline{\theta}^* + F_2(t)\underline{g}(t) \tag{4.31}$$

where

$$F_1(t) = \exp[- S_{B2}^{-1} S_{B4}t]$$

and

$$F_2(t) = \int_0^t \exp[S_{B2}^{-1}S_{B4}(\tau-t)]d\tau \tag{4.32}$$

From Eq. (4.23) we get

$$\dot{\underline{\theta}} = -[S_{B2}]^{-1}[S_{B1}\dot{\underline{\omega}}_B + S_{B3}\underline{\omega}_B + S_{B5}\underline{\omega}_B\delta_{B2}\dot{\underline{\omega}}_B + \dot{\delta}_B^*\omega_B I_B\underline{\omega}_B - \delta_B^*\underline{u}_B] - [S_{B2}]^{-1}S_{B4}\underline{\theta} \tag{4.33}$$

Here $\underline{\theta}$ is given by Eq. (4.25).

Differentiating Eq. (4.25), we get

$$\ddot{\underline{\theta}} = -[S_{B2}]^{-1}[S_{B1}\ddot{\underline{\omega}}_B + S_{B3}\dot{\underline{\omega}}_B + S_{B5}\dot{\omega}_B\delta_{B2}\underline{\omega}_B + S_{B5}\omega_B\delta_{B2}\dot{\underline{\omega}}_B + \dot{\delta}_B^*\omega_B I_B\underline{\omega}_B + \delta_B^*\omega_B I_B\dot{\underline{\omega}}_B - \dot{\delta}_B^*\underline{u}_B] - [S_{B2}]^{-1}S_{B4}\dot{\underline{\theta}} \tag{4.34}$$

Here $\dot{\underline{\theta}}$ is given by Eq. (4.33).

5. <u>Equation for</u> $\underline{\omega}_B$

The variables $\underline{\theta}$, $\dot{\underline{\theta}}$ and $\ddot{\underline{\theta}}$ are now eliminated from Eq. (4.21) to obtain an equation in $\underline{\omega}_B$ only. The values of $\underline{\theta}$, $\dot{\underline{\theta}}$ and $\ddot{\underline{\theta}}$ are taken from Eqs. (4.25), (4.33) and (4.34), respectively. Thus the equation for

$\underline{\omega}_B$ is

$$S_{A1}S_{B2}^{-1}S_{B1}\ddot{\underline{\omega}}_B + [S_{A1}S_{B2}^{-1}S_{B3} - S_{A1} + (S_{A2} + S_{A3} - S_{A1}S_{B2}^{-1}S_{B4})S_{B2}^{-1}]\dot{\underline{\omega}}_B$$

$$+ [(S_{A2} + S_{A3} - S_{A1}S_{B2}^{-1}S_{B4})S_{B2}^{-1} - S_{A2}]\underline{\omega}_B + [2S_{A1}\tilde{\omega}_B + S_{A5}\tilde{\omega}_B\delta_{A2}$$

$$- S_{A5}\widetilde{(\delta_{A2}\underline{\omega}_B)} + \delta_A^*\tilde{\omega}_B I_A - \delta_A^*\widetilde{(I_A\underline{\omega}_B)}]S_{B2}^{-1}(S_{B1}\dot{\underline{\omega}}_B + S_{B3}\underline{\omega}_B$$

$$+ S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B + \delta_B^*\tilde{\omega}_B I_B\underline{\omega}_B - \delta_B^*\underline{u}_B) + (S_{A2} + S_{A3}$$

$$- S_{A1}S_{B2}^{-1}S_{B4})S_{B2}^{-1}(S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B + \delta_B^*\tilde{\omega}_B I_B\underline{\omega}_B - \delta_B^* u_B)$$

$$- \{[2S_{A1}\tilde{\omega}_B + S_{A2} + S_{A3} - S_{A1}S_{B2}^{-1}S_{B4} + S_{A5}\tilde{\omega}_B\delta_{A2} - S_{A5}\widetilde{(\delta_{A2}\underline{\omega}_B)}$$

$$+ \delta_A^*\tilde{\omega}_B I_A - \delta_A^*\widetilde{(I_A\underline{\omega}_B)}]S_{B2}^{-1}S_{B4} + S_{A1}(\tilde{\dot{\omega}}_B + \tilde{\omega}_B\tilde{\omega}_B) + S_{A2}\tilde{\omega}_B$$

$$+ S_{A4} + S_{A5}\tilde{\omega}_B\delta_{A2}\tilde{\omega}_B - S_{A5}\widetilde{(\delta_{A2}\underline{\omega}_B)} + \delta_A^*\tilde{\omega}_B I_A\tilde{\omega}_B$$

$$- \delta_A^*\widetilde{(I_A\underline{\omega}_B)}\tilde{\omega}_B\}\theta - S_{A5}\{\widetilde{(S_{B2}^{-1}S_{B1}\dot{\underline{\omega}}_B)} + \widetilde{(S_{B2}^{-1}S_{B3}\underline{\omega}_B)}$$

$$+ \widetilde{(S_{B2}^{-1}S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B)} + \widetilde{(S_{B2}^{-1}\delta_B^*\tilde{\omega}_B I_B\underline{\omega}_B)} - \widetilde{(S_{B2}^{-1}\delta_B^*\underline{u}_B)}$$

$$+ \widetilde{(S_{B2}^{-1}S_{B4}\underline{\theta})}\}\delta_{A2}\{S_{B2}^{-1}(S_{B1}\dot{\underline{\omega}}_B + S_{B3}\underline{\omega}_B + S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B$$

$$+ \delta_B^*\tilde{\omega}_B I_B\underline{\omega}_B - \delta_B^*\underline{u}_B + S_{B4}\underline{\theta}) - \tilde{\omega}_B\underline{\theta}\}$$

$$- \delta_A^*\{\widetilde{(S_{B2}^{-1}S_{B1}\dot{\underline{\omega}}_B)} + \widetilde{(S_{B2}^{-1}S_{B3}\underline{\omega}_B)} + \widetilde{(S_{B2}^{-1}S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B)}$$

$$+ \widetilde{(S_{B2}^{-1}\delta_B^*\tilde{\omega}_B I_B\underline{\omega}_B)} - \widetilde{(S_{B2}^{-1}\delta_B^*\underline{u}_B)} + \widetilde{(S_{B2}^{-1}S_{B4}\theta)}\}I_A\{S_{B2}^{-1}(S_{B1}\dot{\underline{\omega}}_B$$

$$+ S_{B3}\underline{\omega}_B + S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B + \delta_B^*\tilde{\omega}_B I_B\underline{\omega}_B - \delta_B^*\underline{u}_B + S_{B4}\underline{\theta})$$

$$- \tilde{\omega}_B\underline{\theta}\} + [S_{A5}\widetilde{(\tilde{\omega}_B\underline{\theta})}\delta_{A2} + \delta_A^*\widetilde{(\tilde{\omega}_B\underline{\theta})}I_A]S_{B2}^{-1}(S_{B1}\dot{\underline{\omega}}_B$$

$$+ S_{B3}\underline{\omega}_B + S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B + \delta_B^*\tilde{\omega}_B I_B\underline{\omega}_B - \delta_B^*\underline{u}_B + S_{B4}\underline{\theta})$$

$$- [S_{A5}\widetilde{(\tilde{\omega}_B\underline{\theta})}\delta_{A2}\tilde{\omega}_B\underline{\theta} + \overset{*}{\delta}_A\widetilde{(\tilde{\omega}_B\underline{\theta})}I_A\tilde{\omega}_B\underline{\theta}]$$

$$+ S_{A1}S_{B2}^{-1}(S_{B5}\dot{\tilde{\omega}}_B\delta_{B2}\underline{\omega}_B + S_{B5}\tilde{\omega}_B\delta_{B2}\dot{\underline{\omega}}_B + \overset{*}{\delta}_B\dot{\tilde{\omega}}_B I_B\underline{\omega}_B$$

$$+ \overset{*}{\delta}_B\tilde{\omega}_B I_B\dot{\underline{\omega}}_B) + S_{A5}\tilde{\omega}_B\delta_{A2}\underline{\omega}_B + \overset{*}{\delta}_A\tilde{\omega}_B I_A\underline{\omega}_B$$

$$+ S_{A1}S_{B2}^{-1}\overset{*}{\delta}_B\ddot{\underline{u}}_B - \overset{*}{\delta}_A\underline{u}_A = 0 \tag{4.35}$$

6. Perturbation Analysis

Before making any attempts to solve Eq. (4.35), we look into the rigid body motion of a dual-spin system with constant relative spin rate. The equations for such a system are given by

$$I_1\dot{\omega}_{B,1} + (I_3 - I_2)\omega_{B,2}\omega_{B,3} + I_{A3}\omega_{B2}\dot{\theta}_3 = \varepsilon' F_1' \tag{4.36}$$

$$I_2\dot{\omega}_{B,2} - (I_3 - I_1)\omega_{B,1}\omega_{B,3} - I_{A3}\omega_{B1}\dot{\theta}_3 = \varepsilon' F_2' \tag{4.37}$$

$$I_3\dot{\omega}_{B,3} = 0 \tag{4.38}$$

where

$$I_i = I_{Ai} + I_{Bi} \quad \text{for} \quad i = 1,2,3,$$

and $\varepsilon' F_1'$ and $\varepsilon' F_2'$ are external torques, and ε' is a very small parameter. With zero initial conditions, the zeroth order solution of this system is given by

$$\underline{\omega}_B = 0.$$

Let it be assumed that $I_2 = I_1$. Then the first order solutions are given by

$$\omega_{B,1} = \frac{\varepsilon' F_2'}{I_{A3}\dot{\theta}_3}\left(\cos\frac{I_{A3}\dot{\theta}_3}{I_1}t - 1\right) + \frac{\varepsilon' F_1'}{I_{A3}\dot{\theta}_3}\sin\frac{I_{A3}\dot{\theta}_3}{I_1}t \tag{4.39}$$

$$\omega_{B,2} = \frac{\varepsilon' F_1'}{I_{A3}\dot{\theta}_3}\left(1 - \cos\frac{I_{A3}\dot{\theta}_3}{I_1}t\right) + \frac{\varepsilon' F_2'}{I_{A3}\dot{\theta}_3}\sin\frac{I_{A3}\dot{\theta}_3}{I_1}t \tag{4.40}$$

if

$$\omega_{B,3} = O[(\varepsilon')^2] \tag{4.41}$$

With this analysis in view, we now begin a perturbation analysis of Eq. (4.35) by assuming $\underline{\omega}_B$ in the form

$$\underline{\omega}_B = \underline{x}_0 + \underline{h}$$

where

$$\underline{h} = \varepsilon'\underline{x}_1 + (\varepsilon')^2\underline{x}_2 + (\varepsilon')^3\underline{x}_3 + \dots \tag{4.42}$$

We also take

$$\underline{u}_A = \underline{u}_{A0} + \varepsilon'\underline{u}_{A1} + (\varepsilon')^2\underline{u}_{A2} + (\varepsilon')^3\underline{u}_{A3} + \dots \tag{4.43}$$

$$\underline{u}_B = \underline{u}_{B0} + \varepsilon'\underline{u}_{B1} + (\varepsilon')^2\underline{u}_{B2} + (\varepsilon')^3\underline{u}_{B3} + \dots \tag{4.44}$$

$$\underline{\theta}(\underline{\omega}_B, \dot{\underline{\omega}}_B, t) = F_1(t)\underline{\theta}^* + \int_0^t \{\exp[S_{B2}^{-1}S_{B4}(\tau - t)](\delta_B^*\underline{u}_B - S_{B1}\dot{\underline{\omega}}_B - S_{B3}\underline{\omega}_B)\}d\tau + \varepsilon' F_3(\underline{\omega}_B, t) \tag{4.45}$$

where

$$\varepsilon' F_3(\underline{\omega}_B, t) = -\{\int_0^t \exp[S_{B2}^{-1}S_{B4}(\tau - t)]d\tau\}(S_{B5}\omega_B\delta_{B2} + \delta_B^*\omega_B I_B)\underline{\omega}_B$$

and ε' will be taken as the maximum value of the external torques. Equation (4.45) is obtained from Eqs. (4.25) and (4.26) by separating the linear and nonlinear parts.

Now, separating the linear and nonlinear terms, Eq. (4.35) for $\underline{\omega}_B$ is rewritten as

$$(S_{A1}S_{B2}^{-1}S_{B1})\ddot{\underline{\omega}}_B + [S_{A1}S_{B2}^{-1}S_{B3} - S_{A1} + (S_{A2} + S_{A3} - S_{A1}S_{B2}^{-1}S_{B4})S_{B2}^{-1}]\dot{\underline{\omega}}_B$$
$$+ [(S_{A2} + S_{A3} - S_{A1}S_{B2}^{-1}S_{B4})S_{B2}^{-1} - S_{A2}]\underline{\omega}_B - [(S_{A2} + S_{A3}$$
$$- S_{A1}S_{B2}^{-1}S_{B4})S_{B2}^{-1}S_{B4} + S_{A4}]\{F_1(t)\underline{\theta}^* + \int_0^t \exp[S_{B2}^{-1}S_{B4}(\tau - t)](\delta_B^*\underline{u}_B$$

$$- S_{B1}\dot{\underline{\omega}}_B - S_{B3}\underline{\omega}_B)d\tau\} + S_{A1}S_{B2}^{-1}\delta_B^{*}\dot{\underline{u}}_B$$

$$- [(S_{A2} + S_{A3} - S_{A1}S_{B2}^{-1}S_{B4})S_{B2}^{-1}\delta_B^{*}]\underline{u}_B - \delta_A^{*}\underline{u}_A$$

$$+ \varepsilon'\underline{f}(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{\theta}, \underline{u}_A, \underline{u}_B) = 0 \qquad (4.46)$$

where $\varepsilon'\underline{f}$ consists of all the remaining nonlinear terms in Eq. (4.35). Defining $B_1, B_2, \ldots, B_7$ as the coefficients of Eq. (4.46), this equation takes the form

$$B_1\ddot{\underline{\omega}}_B + B_2\dot{\underline{\omega}}_B + B_3\underline{\omega}_B + B_4\ \{\int_0^t B_5(\tau - t)[\delta_B^{*}\underline{u}_B$$

$$- S_{B1}\dot{\underline{\omega}}_B - S_{B3}\underline{\omega}_B](\tau)d\tau\} + B_6\underline{u}_B + B_7\underline{u}_B - \delta_A^{*}u_A$$

$$+ \varepsilon'\underline{f}(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B) = - B_4F_1\underline{\theta}^{*} \qquad (4.46a)$$

Expanding $\underline{f}(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B)$ about $\underline{x}_0, \dot{\underline{x}}_0, \underline{u}_{A0}$ and $\underline{u}_{B0}$, in a Taylor's series, we obtain the following expansion:

$$f_i(v) = f_i(\underline{v}_0) + f_{i,j}(\underline{v}_0)v_j' + \frac{1}{2} f_{i,jk}v_j'v_k' + \ldots \qquad (4.47)$$

where

$$\underline{v} = [\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B]^T \qquad (4.48)$$

$$= \sum_{i=0}^{\infty} (\varepsilon')^i \underline{v}_i$$

and

$$\underline{v}' = \underline{v} - \underline{v}_0$$

such that

$$\underline{v}_i = [\dot{\underline{x}}_i, \underline{x}_i, \underline{u}_{Ai}, \underline{u}_{Bi}]^T \qquad (4.49)$$

From Eq. (4.46a), the coefficients of $(\varepsilon')^0$, ε', $(\varepsilon')^2$ etc. generate the following equations:

$$B_1\ddot{\underline{x}}_0 + B_2\dot{\underline{x}}_0 + B_3\underline{x}_0 - B_4 \int_0^t [B_5(\tau - t)(S_{B1}\dot{\underline{x}}_0 + S_{B3}\underline{x}_0)(\tau)]d\tau$$

$$+ \underline{w}_0 = - B_4F_1\underline{\theta}^* \quad (4.50)$$

$$B_1\ddot{\underline{x}}_1 + B_2\dot{\underline{x}}_1 + B_3\underline{x}_1 - B_4 \int_0^t [B_5(\tau - t)(S_{B1}\dot{\underline{x}}_1 + S_{B3}\underline{x}_1)(\tau)]d\tau$$

$$+ \underline{w}_1 = - \underline{f}(\underline{v}_0) \quad (4.51)$$

$$B_1\ddot{\underline{x}}_2 + B_2\dot{\underline{x}}_2 + B_3\underline{x}_2 - B_4 \int_0^t [B_5(\tau - t)(S_{B1}\dot{\underline{x}}_1 + S_{B3}\underline{x}_2)(\tau)]d\tau$$

$$+ \underline{w}_2 = - \frac{\partial \underline{f}}{\partial \dot{\underline{\omega}}_B}(\underline{v}_0)\dot{\underline{x}}_1 - \frac{\partial \underline{f}}{\partial \underline{\omega}_B}(\underline{v}_0)\underline{x}_1 - \frac{\partial \underline{f}}{\partial \underline{u}_A}(\underline{v}_0)\underline{u}_{A1} - \frac{\partial \underline{f}}{\partial \underline{u}_B}(\underline{v}_0)\underline{u}_{B1}$$

$$(4.52)$$

In these equations,

$$\underline{w}_i = B_4 \int_0^t [B_5(\tau - t)\overset{*}{\delta}_B\underline{u}_{Bi}(\tau)]d\tau + B_6\dot{\underline{u}}_{Bi} + B_7\underline{u}_{Bi} - \overset{*}{\delta}_A\underline{u}_{Ai} \quad (4.53)$$

It is thus seen from Eqs. (4.50) through (4.53), that all the variables in this problem are given by linear integrodifferential equations. To solve these equations the functions w_i are to be explicitly known in terms of $\underline{\omega}_B$, $\dot{\underline{\omega}}_B$ and time.

7. Formal Solutions

Equation (4.50) is now formally solved as follows:

The variables $\underline{x}_0$ and $\dot{\underline{x}}_0$ are assumed to remain close to zero, so that these variables are replaced by their initial values under the integral sign. If $\phi_1(t)$ and $\phi_2(t)$ be the fundamental matrices of the homogeneous system

$$B_1\ddot{\underline{x}}_0 + B_2\dot{\underline{x}}_0 + B_3\underline{x}_0 = 0$$

such that the homogeneous solution is given by

$$\underline{x}_0(t) = \phi_1\underline{x}_0(0) + \phi_2\dot{\underline{x}}_0(0),$$

then the complete solution to the inhomogeneous system is given by

$$\underline{x}_0 = \phi_1(t)\underline{x}_0(0) + \phi_2(t)\dot{\underline{x}}_0(0) - \int_0^t \phi_2(t - \tau)B_4[F_1(\tau)\underline{\theta}^*$$

$$- \int_0^\tau B_5(s-\tau)(S_{B1}\dot{\underline{x}}_0(0) + S_{B3}\underline{x}_0(0))ds]d\tau - \int_0^t \phi_2(t - \tau)\underline{w}_0(\tau)d\tau$$

(4.54)

In a similar way, Eqs. (4.51) and (4.52) are solved to get

$$\underline{x}_1 = \phi_1(t)\underline{x}_1(0) + \phi_2(t)\dot{\underline{x}}_1(0) + \int_0^t \phi_2(t - \tau)\{B_4 \int_0^\tau B_5(s - \tau)[S_{B1}\dot{\underline{x}}_1(0)$$

$$+ S_{B3}\underline{x}_1(0)]ds - \underline{w}_1(\tau) - f_1(\underline{v}_0(\tau))\}d\tau \qquad (4.55)$$

$$\underline{x}_2 = \phi_1(t)\underline{x}_2(0) + \phi_2(t)\dot{\underline{x}}_2(0) + \int_0^t \phi_2(t - \tau)\{B_4 \int_0^\tau B_5(s - \tau)[S_{B1}\dot{\underline{x}}_2(0)$$

$$+ S_{B3}\underline{x}_2(0)]ds - [\underline{w}_2(\tau) + \underline{\nabla f}(v_0)\underline{v}_1](\tau)\}d\tau \qquad (4.56)$$

Some of the initial conditions are obtained by setting the secular terms equal to zero. The functions $\underline{x}_3$, $\underline{x}_4$, etc. will be obtained similarly.

8. Control Torques

It has been mentioned that the angular velocities can be solved when the control torque functions $\underline{w}_i$ are known. Instead of choosing $\underline{w}_i$ arbitrarily, the forms of $\underline{w}_i$ as functions of $\underline{\omega}_B$ are now obtained by an optimum control policy. The necessary theoretical considerations are briefly mentioned here. Further theoretical details are given in [1] and [2].

Let E denote the Banach space of triplets $[u_1(t), u_2(t), u_3(t)]$ where $u_i(t)$ belong to $L_1(0,T)$. The endpoint T of the time interval is considered to be fixed here, and $T > 0$. The norm of the vector $\underline{u}(t) = \{u_i(t)\}$ in E is

$$\|u\| = \sum_{i=1}^{3} \int_0^T |u_i(t)|dt$$

The dual space E^* then will consist of all triplets $\underline{w} = [w_1(t), w_2(t), w_3(t)]$ with $w_i(t)$ in $L_\infty(0,T)$ such that for every $\underline{w}(t)$ in E^*, we have

$$w = \max_{\substack{1 \leq j \leq 3 \\ 0 \leq t \leq T}} w_j(t) \ .$$

Then, for every $\underline{u}$ in E and $\underline{w}$ in E^*, the inner product is defined as

$$u,w = \sum_{j=1}^{3} \int_0^T u_j(t)w_j(t)dt \ .$$

In this problem the vector w_j in E^* will be used to denote the control torque vectors $\underline{w}_i$ given in Eq. (4.53), and the variables $\underline{x}_i$ are assumed to be in the space E.

Let Eq. (4.54) be considered now. The assumed time optimal control policy is to choose $\underline{w}_0$ such that

$$\underline{x}_0(T) = \dot{\underline{x}}_0(T) = 0 \tag{4.57}$$

and

$$\|w_0\| \leq M_0 \tag{4.58}$$

for the minimum time T.

Equation (4.54) can be expressed as

$$x_{0,i}(t) = x^*_{0,i}(t) - \int_0^t \phi_{2,ij}(t - \tau)w_{0j}(\tau)d\tau \tag{4.59}$$

where

$$\underline{x}_0^* = \phi_1(t)\underline{x}_0(0) + \phi_2(t)\dot{\underline{x}}_0(0) - \int_0^t \phi_2(t - \tau)B_4[F_1(\tau)\underline{\theta}^*$$

$$- \int_0^T B_5(s - \tau)(S_{B1}\dot{\underline{x}}_0(0) + S_{B3}\underline{x}_0(0))ds]d\tau \ . \tag{4.60}$$

Differentiating Eq. (4.59), we obtain

$$\dot{\underline{x}}_0(t) = \dot{\underline{x}}_0^*(t) - \int_0^t \phi_2(t - \tau)w_{0j}(\tau)d\tau \tag{4.61}$$

From Eqs. (4.59) and (4.61), the requirements given by Eq. (4.57) become

$$0 = \underline{x}_0^*(T) - \langle \phi_2, \underline{w}_0 \rangle \tag{4.62}$$

and

$$0 = \dot{\underline{x}}_0^*(T) - \langle \dot{\phi}_2, \underline{w}_0 \rangle \tag{4.63}$$

With the above equations, for a fixed $T > 0$, the optimum control problem is to find the function $\underline{w}_0$ in E^* such that $\|\underline{w}_0\|$ is minimum and Eqs. (4.62) and (4.63) are satisfied. If, for each $T > 0$, the smallest $\|w_0\| > M_0$, then the problem has no solution, and the allowable value of M_0 has to be increased.

As ϕ_2 and $\dot{\phi}_2$ are linearly independent, a Cartesian subspace F of E can be defined which consists of the elements of ϕ_2 and $\dot{\phi}_2$. Let $\underline{w}_0^*$ be the linear functional on F determined by Eqs. (4.62) and (4.63). From the Hahn-Banach theorem, we know that the functional $\underline{w}_0^*$ can be extended to all E, and hence to E^*, and the resulting functional will have the same norm as that of $\underline{w}_0^*$. Therefore

$$M_0 = \|\underline{w}_0\| = \|\underline{w}_0^*\| = \max_{\psi \text{ in } F} \frac{\langle \psi, \underline{w}_0^* \rangle}{\|\psi\|}$$

or

$$M_0 = \max_{\underline{\mu}_1} \frac{\underline{\mu}_1 \langle \psi, \underline{w}_0^* \rangle}{\|\underline{\mu}_1 \cdot \psi\|} \tag{4.65}$$

where

$$\psi = [\phi_2, \dot{\phi}_2]^T \tag{4.65a}$$

Let it be assumed that for some particular values of the multiplier $\underline{\mu}_1$, the maximum is obtained in Eq. (4.65). Then from Eq. (4.65), we obtain

$$M_0 \| \underline{\mu}_1 \cdot \psi \| = \int_0^T \underline{\mu}_1 [\psi(T - s)] \underline{w}_0(s) ds$$

$$\leq M_0 \int_0^T | \underline{\mu}_1 \cdot \psi(T - s) | ds \qquad (4.66)$$

Since $\underline{\mu}_1$ provides the maximum in Eq. (4.65), the last two quantites in Eq. (4.66) must be equal. This is possible only if

$$\underline{w}_0(t) = M_0 \operatorname{sgn} \{\mu_1 \cdot \psi(T - t)\} \qquad (4.67)$$

Equation (4.67) together with Eqs. (4.62) and (4.63) provide the functional $\underline{w}_0$, and it is seen that this is a "bang-bang" control in $\underline{w}_0$.

The maximization problem of Eq. (4.65) becomes the minimization problem of Λ_1, subject to the constraint

$$\Lambda_2 = 0$$

where

$$\Lambda_1 = \| \mu_1 \cdot \psi \| \qquad (4.68)$$

and

$$\Lambda_2 = \mu_1 \cdot \langle \psi, \underline{w}_0 \rangle - 1 = 0 \qquad (4.69)$$

Also, from Eqs. (4.62) and (4.63), we have

$$\langle \psi, \underline{w}_0 \rangle = [x_0^*(T), \dot{x}_0^*(T)]^T \quad . \qquad (4.70)$$

Thus the problem is to minimize the integral

$$G = \int_0^T \{ \sum_j | \sum_i \mu_{1i} \psi_{ij}(t) | - \mu_2 \Lambda_2 \} dt \qquad (4.71)$$

where μ_2 is a scalar Lagrangian multiplier. The integrand in Eq. (4.71) is only a linear function in μ_1, so the minimum is obtained by the traversality conditions only at $t = T$, [3]. Hence we solve

$$\Lambda_2(T) = 0 \tag{4.72}$$

and

$$\mu_{1i}\psi_{ij}(T) = 0, \ (j = 1,2,3; \ i = 1\text{-}6) \ . \tag{4.73}$$

Since ψ is given by Eq. (4.65a) where ϕ_2 and $\dot{\phi}_2$ are independent functions, the Eq. (4.73) is decomposed into

$$\mu_{1i}\psi_{ij}(T) = 0; \quad (i,j = 1\text{-}3) \tag{4.74}$$

and

$$\mu_{1i}\psi_{ij}(T) = 0; \quad (j = 1\text{-}3; \ i = 4\text{-}6) \tag{4.75}$$

The six equations of (4.74) and (4.75) and the scalar equation (4.72) determine the seven variables μ_{1i}, $i = 1\text{-}6$ and T.

So far only the Eq. (4.54) has been considered after reducing it to the form of Eq. (4.59). Since Eqs. (4.55) and (4.56) can also be reduced to the form of Eq. (4.59), the components $\underline{w}_1$, $\underline{w}_2$ etc. will be obtained by a similar procedure.

Thus, from Eqs. (4.53) and (4.67), we obtain

$$B_4 \int_0^t B_5(\tau - t)\overset{*}{\delta}_{\underline{B}}\underline{u}_{Bi}(\tau)d\tau + B_{\underline{6}}\dot{\underline{u}}_{Bi} + B_7 u_{Bi} - \overset{*}{\delta}_A u_{Ai}$$

$$= M_i \ \text{sgn}[\underline{\mu}_1 \cdot \psi(T - t)] \tag{4.74}$$

Here one of the vectors $\underline{u}_{Ai}$ and $\underline{u}_{Bi}$ can be chosen arbitrarily. Thus we consider the body A to be completely uncontrolled. Then from Eqs. (4.22) and (4.43), we have

$$\underline{u}_{A0} = 0 \tag{4.75}$$

$$\underline{u}_{A1} = \frac{1}{\varepsilon'} \underline{T}_{EA}(\omega_{A0}) \tag{4.76}$$

$$\underline{u}_{A2} = \frac{1}{\varepsilon'} \frac{\partial}{\partial \underline{\omega}_A} \underline{T}_{EA}(\omega_{A0})\omega_{A1} \tag{4.77}$$

where

$$\underline{\omega}_A = \omega_{A0} + \varepsilon'\omega_{A1} + (\varepsilon')^2\omega_{A2} + \dots \tag{4.78}$$

Equation (4.74) is a linear integrodifferential equation and will be solved by known methods.

References

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