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Report file

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TO: Contracting Officer, Code 245, NASA/GSFC
Technical Officer, Code 651, NASA/GSFC

FROM: Thomas O. Haig
Executive Director

REFERENCE: Contract NAS5-21798

SUBJECT: Monthly Progress Report for "Studies of Soundings and Imaging Measurements from Geostationary Satellites"

Task A Investigation of Meteorological Data Processing Techniques

Since much of our work depends upon the efforts of graduate students, there is always a reduction of level of effort over the end-of-semester, holiday season. The work will pick up during January.

The earth-edge determination algorithm results have been compared with navigation derived from landmarks. It was found that apparent E-W landmark "motion," i.e., navigation inaccuracy, corresponded closely to the apparent motion of the earth edge at the landmark latitude within one to two ATS data samples--(1.35 RMS) which is less than one IFOV. These results are very encouraging and indicate that our algorithm may be successful in correcting line start jitter. We are developing an implementation scheme which may remove this persistent error source.

Task B Sun Glitter

This work was suspended during December because of final examination conflict.

Task D Cloud Growth Rate

Some progress has been made in the ATS III data analysis since the last report. However, estimation of cirrus shield thickness directly from the cloud brightness data seems difficult because such a relationship not only depends on sun-satellite-cloud geometry, but also on cloud particle composition (water droplets or ice crystals) and their spatial

Task D Cloud Growth Rate (Continued)

distribution. An attempt is being made to develop a model which will relate these parameters for a homogeneous plane cloud composed of either water droplets or ice crystals for a given size distribution. This model may help explain some of the unresolved satellite observed severe storm characteristics. A graduate student is working on this problem.

Task E Comparative Studies in Satellite Stability

Part V of the results of this study is attached. With completion of this portion of the analysis, work will now proceed rapidly in coding the satellite stability model for the 1108 Computer. Approximately two months will be required to complete the computer model. Mr. Das has developed an approach which will permit the model to be run on the 1108 without compromise to the model's generality yet without excessive computer time being required.

TOH/jz

Enclosure: PART V. Random Analysis of the Prediction Problem

PART V

RANDOM ANALYSIS OF THE PREDICTION PROBLEM

Contents

Nomenclature

P_0	= Expectation matrix for $\underline{q}$ [Eq. (5.5)]
$\underline{z}_1$	= Angular velocity sensor data
$\bar{\underline{z}}_1$	= Mean value of $\underline{z}_1$
$R_1(t)$	= Expectation matrix for z_1 [Eq. (5.9)]
$\bar{\underline{u}}_A$, $\bar{\underline{u}}_B$, $\bar{\underline{w}}$	= Mean values of $\underline{u}_A$, $\underline{u}_B$ and $\underline{w}$
U_A, U_B	= Expectation matrices for $\underline{u}_A, \underline{u}_B$ [Eqs. (5.12) and (5.13)]
ρ_{ji}	= Transformed error variables [Eqs. (5.14) through (5.17)]
R_0	= Expectation matrix for $\omega_B(0)$ [Eq. (5.18)]
$\underline{\lambda}_1(t)$	= Lagrangian vector multipliers [Eq. (5.24)]
$\phi_1(t)$, $\phi_2(t)$	= Fundamental matrices of Eq. (5.58)
$\underline{q}_0$, $\underline{q}_{ij}$, $\underline{q}_{ijk}$	= Components of $\underline{q}$ [Eq. (5.89)]
A_{ijk}	= Components of A_i [Eqs. (5.87), (5.88), (5.112)]
$\psi_1(t)$, $\psi_2(t)$	= Fundamental matrices of Eq. (5.90)
$\underline{\psi}_{Bi}(t)$	= Angular error for the body B [Eq. (5.127)]
$P_{Bi}(t)$	= Expectation matrix for $\underline{\psi}_{Bi}$]Eq. (5.129)]
$\underline{\psi}_{Ai}(t)$	= Angular error for the body A [Eq. (5.131)]
$P_{Ai}(t)$	= Expectation matrix for $\underline{\psi}_{Ai}$ [Eq. (5.132)]

1. Introduction

A purely deterministic analysis has been made in the previous parts of this work. Now the effects of random forcing and control torques and measurement errors will be considered.

It has been derived in Eq. (4.46) that the angular velocity $\underline{\omega}_B$ is given by an equation of the form

$$\ddot{\underline{\omega}}_B + A\dot{\underline{\omega}}_B = f_1(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B, \underline{\theta}^*, t) \tag{5.1}$$

where $\underline{u}_B$, $\underline{u}_B$ are functions of external and control torques and $\underline{\theta}^*$ is the initial value of the rotation of the body A relative to B. Equation (5.1) gives $\underline{\omega}_B$ in the form

$$\underline{\omega}_B = f_2(\dot{\underline{\omega}}_B(0), \underline{\omega}_B(0), \underline{u}_A, \underline{u}_B, \underline{\theta}^*, t) \tag{5.2}$$

Since, in eq. (5.2), the values of $\underline{u}_A$, $\underline{u}_B$ are not exactly known, these are to be considered as random variables with certain means and variances. The initial values, $\dot{\underline{\omega}}_B(0)$ and $\underline{\omega}_B(0)$ are also random quantities because

a) these are measured by instruments with inherent error, and

b) the generalized position coordinates of the measuring instruments are also random variables. Thus, from Eq. (5.2), it is seen that $\underline{\omega}_B$ is a random variable.

The position coordinates $\underline{q}_B$ of the structural elements of the satellite are random variables because these are associated with $\underline{\omega}_B$ by Eqs. (1.116) and (1.117) in the form

$$\ddot{\underline{q}} + B\,\dot{\underline{q}} = f_3(\dot{\underline{q}}, \underline{q}, \underline{\omega}_B, \underline{\omega}_A, t) \tag{5.3}$$

Integrating Eq. (5.3), we obtain $\underline{q}$ in the form

$$\underline{q} = f_4[\dot{\underline{q}}(0), \underline{q}(0), \underline{\omega}_B, \underline{\omega}_A, t] \tag{5.4}$$

From Eq. (5.4), it is seen that indeterminacy in the functions $\underline{q}(t)$ is also introduced through the quantities $\dot{\underline{q}}(0)$ and $\underline{q}(0)$ which cannot be measured.

These considerations show that the predicted angular positions of the Earth Viewing Modules on the satellite will also be random quantities. In this part of our analysis the method of obtaining the most likely estimates of these angular positions and the associated probability distribution functions is explained. The necessary theory will be briefly presented here. References [1] and [2] may be consulted for more detailed treatment.

2. Maximal Probabilistic Formulation

It has been mentioned above that it is not possible to obtain the probability distributions of the initial values $\underline{q}_B(0)$ and $\dot{\underline{q}}_B(0)$. To begin the analysis, these distributions will be assumed to be gaussian with zero mean and known variances. So the function P_0 is considered to be known, where

$$P_0 = E\{[\underline{q}(0)]^T[\underline{q}(0)]\} . \tag{5.5}$$

Here E denotes statistical expectation. Estimates of P_0 will be refined by an iteration process after obtaining Eq. (5.4).

Let $\underline{z}_1(t)$ be the data transmitted by sensors on the satellite for calculating the initial angular velocities of the satellite. Let $\underline{v}_1(t)$ and $\underline{v}_2[\underline{q}(0)]$ be the errors associated with $\underline{z}_1(t)$ due to the inherent errors of the sensor data transmission and the motion of sensors respectively. Then

$$\underline{z}_1(t) = \overline{\underline{z}}_1(t) + \underline{v}_1(t) + \underline{v}_2[\underline{q}(0)] \tag{5.6}$$

where $\bar{z}_1(t)$ is the correct data that would have been obtained with a perfect sensor mounted on an absolutely rigid satellite. It is assumed that $\underline{v}_1(t)$ is gaussian with zero mean and known variance. As $\underline{v}_2$ is of the form

$$v_{2,i} = c^*_{ij}\, q_j(0), \tag{5.7}$$

so $\underline{v}_2[\underline{q}(0)]$ is also gaussian with zero mean and known variance. Let the $\bar{\underline{w}}_B(0)$, the mean values of $\underline{w}_B(0)$, be given by the relation

$$\bar{\underline{z}}_1(t) = \underline{h}_1[\bar{\underline{\omega}}_B(0), t]. \tag{5.8}$$

Let us also assume that the error signals $(\underline{v}_1 + \underline{v}_2)$ at different times are not correlated. So the expectation of $(\underline{v}_1 + \underline{v}_2)$ are obtained in the form

$$E\{[\underline{z}_1(t) - \bar{\underline{z}}_1(t)]\ [\underline{z}_1(\tau) - \bar{\underline{z}}_1(\tau)]^T\} = R_1(\tau)\delta(t-\tau) \tag{5.9}$$

The distribution of $\underline{v}_1$ is known. With the assumed distribution of $\underline{q}(0)$, the distribution of $\underline{v}_2$ is given by Eq. (5.7). Hence, $R_1(t)$ is calculated from Eq. (5.9).

Let $\bar{\underline{u}}_A(t)$ and $\bar{\underline{u}}_B(t)$ be the theoretical torque functions given by equations of the form

$$\bar{\underline{u}}_A(t) = \underline{g}_A(\underline{\omega}_A, t) \tag{5.10}$$

and

$$\bar{\underline{u}}_B(t) = \underline{g}_B(\underline{\omega}_B, t). \tag{5.11}$$

But due to uncertainties in the external torques and mechanical imperfections, there will be some error associated with $\bar{\underline{u}}_A$ and $\bar{\underline{u}}_B$. These errors are assumed to be gaussian with zero means and known variances, and uncorrelated at different instants of time. So the functions U_A and U_B are assumed to be known, when these are given by

$$E\{[\underline{u}_A(t) - \bar{\underline{u}}_A(t)]\ [\underline{u}_A(\tau) - \bar{\underline{u}}_A(\tau)]^T\} = U_A(t)\cdot\delta(t-\tau) \qquad (5.12)$$

$$E\{[\underline{u}_B(t) - \bar{\underline{u}}_B(t)]\ [\underline{u}_B(\tau) - \bar{\underline{u}}_B(\tau)]^T\} = U_B(t)\cdot\delta(t-\tau) \qquad (5.13)$$

Let $0 \leq t \leq T$ be the domain of this analysis. Let t_1, t_2, t_3 etc. be intermediate points of the interval $(0,T)$, such that

$$0 < t_1 < t_2 < \ . \ . \ . < t_N = T$$

and

$$t_i - t_{i-1} = \Delta t \qquad \text{for all } i.$$

Let ρ_{ji}, $j = 0 - 3$, be defined by

$$\rho_{0i} = \underline{\omega}_B(0) - \bar{\underline{\omega}}_B(0) \qquad (5.14)$$

$$\rho_{1i} = \int_{t_{i-1}}^{t_i} [\underline{u}_A(t) - \bar{\underline{u}}_A(t)]dt \qquad (5.15)$$

$$\rho_{2i} = \int_{t_{i-1}}^{t_i} [\underline{u}_B(t) - \bar{\underline{u}}_B(t)]dt \qquad (5.16)$$

$$\rho_{3i} = \int_{t_{i-1}}^{t_i} [\underline{z}_1(t) - \bar{\underline{z}}_1(t)]dt \qquad (5.17)$$

With these definitions, ρ_{ji} become gaussian with zero means. From Eq. (5.14), we have

$$E\{[\underline{\omega}_B(0) - \bar{\underline{\omega}}_B(0)]\ [\underline{\omega}_B(0) - \bar{\underline{\omega}}_B(0)]^T\}$$

$$= E[\{\underline{h}_1^{-1}[z_1(0)]-h^{-1}[\bar{z}_1(0)]\}\{h_1^{-1}[\underline{z}_1(0)]-h_1^{-1}[\bar{\underline{z}}_1(0)]\}^T]$$

$$= R_0 \qquad (5.18)$$

where R_0 is a known function of $R_1(t)$ introduced in Eq. (5.9). From Eqs. (5.15 through (5.17), we also have

$$E[\rho_{1i}\rho_{1j}^T] = \delta_{ij}\int_{t_{i-1}}^{t_i} U_A(t)\,dt = U_A(t_i)\Delta t.\delta_{ij} \tag{5.19}$$

$$E[\rho_{2i}\rho_{2j}^T] = U_B(t_i)\Delta t\,\delta_{ij} \tag{5.20}$$

$$E[\rho_{3i}\rho_{3j}^T] = R_1(t_i)\Delta t.\delta_{ij}. \tag{5.21}$$

Multiplying together all the individual probability densities of ρ_{ji}, $i = 1 - N$, $j = 0 - 3$, the joint probability density, P, of ρ_{ji} over the whole interval $(0,T)$ is obtained as

$$P = P_0 \exp\left[-\frac{1}{2}\left\{\sum_{i=1}^{N}[\rho_{1i}^T U_A^{-1}(t_i)\rho_{1i} + \rho_{2i}^T U_B^{-1}(t_i)\rho_{2i} + \rho_{3i}^T R_1^{-1}(t_i)\rho_{3i}] + \rho_{0i}R_o^{-1}\rho_{0i}\right\}\right] \tag{5.22}$$

where P_0 is the appropriate normalizing constant.

To obtain the most likely estimates of the variables involved, the density P has to be maximized. For P to be maximum, the argument of the exponential inside the braces has to be minimum. So replacing the summations by integrals and using Eqs. (5.14) through (5.17), the function J to be minimized is obtained as

$$\begin{aligned} J = {} & [\underline{\omega}_B(0) - \bar{\underline{\omega}}_B(0)]^T R_0^{-1}[\underline{\omega}_B(0)-\bar{\underline{\omega}}_B(0)] \\ & + \int_0^T \Big\{[\underline{z}_1(t)-\bar{\underline{z}}_1(t)]^T R_1^{-1}(t)[\underline{z}_1(t)-\bar{\underline{z}}_1(t)] \\ & + [\underline{u}_A(t)-\bar{\underline{u}}_A(t)]^T U_A^{-}(t)[\underline{u}_A(t)-\bar{\underline{u}}_A(t)] \\ & + [\underline{u}_B(t)-\bar{\underline{u}}_B(t)]^T U_B^{-1}(t)[\underline{u}_B(t)-\bar{\underline{u}}_B(t)]\Big\}\,dt \end{aligned} \tag{5.23}$$

To minimize J, Eqs. (5.1), (5.8), (5.10) and (5.11) have to be used as constraints. This means that the functional to be minimized is $(J + J_1)$ where J_1 is given by

$$J_1 = 2\int_0^T \{\underline{\lambda}_1[\ddot{\underline{\omega}}_B + A\dot{\underline{\omega}}_B - f_1(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B, \underline{\theta}^*, t)]$$

$$+ \underline{\lambda}_2[\underline{u}_A - \underline{g}_A] + \underline{\lambda}_3[\underline{u}_B - \underline{g}_B]\} \, dt + 2.\underline{\lambda}_4[\overline{\underline{z}}_1 - \underline{h}_1] \tag{5.24}$$

In Eq. (5.24), $\underline{\lambda}_i$, $i = 1\text{-}4$, are Lagrangian vector multipliers. Now, applying variational methods, the following set of equations are obtained, which are to be solved simultaneously:

$$\ddot{\underline{\omega}}_B + A\dot{\underline{\omega}}_B = f_1(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B, \underline{\theta}^*, t) \tag{5.1}$$

$$\overline{\underline{z}}_1(t) = \underline{h}_1[\underline{\omega}_B(0), t] \tag{5.8}$$

$$\overline{\underline{u}}_A(t) = g_A(\underline{\omega}_A, t) \tag{5.10}$$

$$\overline{\underline{u}}_B(t) = \overline{\underline{g}}_B(\underline{\omega}_B, t) \tag{5.11}$$

$$\underline{\omega}_B(0) = \overline{\underline{\omega}}_B(0) + R_0[A^T\underline{\lambda}_1(0) + \lambda_4] \tag{5.25}$$

$$\underline{u}_A(t) = \overline{\underline{u}}_A(t) + U_A \left(\frac{\partial \underline{f}_1}{\partial \underline{u}_A}\right)^T \underline{\lambda}_1(t) \tag{5.26}$$

$$\underline{u}_B(t) = \overline{\underline{u}}_B(t) + U_B \left(\frac{\partial \underline{f}_1}{\partial \underline{u}_B} - \frac{d}{dt}\frac{\partial \underline{f}_1}{\partial \dot{\underline{u}}_B}\right)^T \underline{\lambda}_1(t) \tag{5.27}$$

$$\ddot{\underline{\lambda}}_1(t) = \left(A - \frac{\partial \underline{f}_1}{\partial \dot{\underline{\omega}}_B}\right)^T \dot{\underline{\lambda}}_1 + \left[\frac{\partial \underline{f}_1}{\partial \underline{\omega}_B} - \frac{d}{dt}\left(\frac{\partial \underline{f}_1}{\partial \dot{\underline{\omega}}_B}\right)\right]^T \underline{\lambda}_1$$

$$+ \left(\frac{\partial \underline{g}_A}{\partial \underline{\omega}_B}\right)^T \cdot \underline{\lambda}_2 + \left(\frac{\partial \underline{g}_B}{\partial \underline{\omega}_B}\right)^T \underline{\lambda}_3 \tag{5.28}$$

$$\underline{u}_A(t) = \overline{\underline{u}}_A(t) + U_A(t).\underline{\lambda}_2 \tag{5.29}$$

$$\underline{u}_B(t) = \overline{\underline{u}}_B(t) + U_B(t)\ \underline{\lambda}_3 \tag{5.30}$$

$$\underline{z}_1(t) = \overline{\underline{z}}_1(t) + R_1(t)\ \underline{\lambda}_4 \tag{5.31}$$

$$\dot{\underline{\lambda}}_1(T) = A^T \underline{\lambda}_1(T) \tag{5.32}$$

The above sets of equations are reduced as follows:

Let $\underline{z}_1(t)$ be the values of $\underline{\omega}_B(0)$ as obtained from the sensor data. Therefore

$$h_1[\underline{\omega}_B(0),\ t] = \underline{\omega}_B(0) \tag{5.33}$$

Hence comparing Eqs. (5.9) and (5.18), we get

$$R_0 = R_1(0) \tag{5.34}$$

So this means that we can disregard the constraint equations (5.8) and (5.31) and also set

$$\underline{\lambda}_4 = 0.$$

From Eqs. (5.26) and (5.29), we get

$$\underline{\lambda}_2 = \left(\frac{\partial \underline{f}_1}{\partial \underline{u}_A}\right)^T \underline{\lambda}_1 \tag{5.35}$$

Similarly, from Eqs. (5.27) and (5.30), we get

$$\underline{\lambda}_3 = \left(\frac{\partial \underline{f}_1}{\partial \underline{u}_B} - \frac{d}{dt}\frac{\partial \underline{f}_1}{\partial \dot{\underline{u}}_B}\right)^T \underline{\lambda}_1 \tag{5.36}$$

Substituting Eqs. (5.35) and (5.36) in Eq. (5.28), we get

$$\ddot{\underline{\lambda}}_1 - \left(A - \frac{\partial \underline{f}_1}{\partial \dot{\underline{\omega}}_B}\right)^T \dot{\underline{\lambda}}_1 - \left[\frac{\partial \underline{f}_1}{\partial \underline{\omega}_B} - \frac{d}{dt}\left(\frac{\partial \underline{f}_1}{\partial \dot{\underline{\omega}}_B}\right) + \left(\frac{\partial \underline{f}_1}{\partial \underline{u}_A}\frac{\partial \underline{g}_A}{\partial \underline{\omega}_B}\right)\right.$$

$$+ \left(\frac{\partial \underline{f}_1}{\partial \underline{u}_B} - \frac{d}{dt} \frac{\partial \underline{f}_1}{\partial \dot{\underline{u}}_B}\right) \frac{\partial \underline{g}_B}{\partial \underline{\omega}_B}\Big]^T \underline{\lambda}_1 = 0 \tag{5.37}$$

From Eqs. (5.10), (5.11), (5.26) and (5.27), Eq. (5.1) becomes

$$\ddot{\underline{\omega}}_B + A \dot{\underline{\omega}}_B - f_1\{\dot{\underline{\omega}}_B, \underline{\omega}_B, [\underline{g}_A + U_A (\frac{\partial \underline{f}_1}{\partial \underline{u}_A})^T \underline{\lambda}_1],$$

$$[\underline{g}_B + U_B (\frac{\partial \underline{f}_1}{\partial \underline{u}_B} - \frac{d}{dt} \frac{\partial \underline{f}_1}{\partial \dot{\underline{u}}_B})^T \underline{\lambda}_1], \underline{\theta}^*, t\} = 0 \tag{5.38}$$

The boundary conditions for $\underline{\lambda}_1$ are given by

$$\dot{\underline{\lambda}}_1(T) = A^T \underline{\lambda}_1(T) \tag{5.32}$$

and

$$\underline{\lambda}_1(0) = R_0^{-1}(A^T)^{-1}[\underline{\omega}_B(0) - \overline{\underline{\omega}}_B(0)] \tag{5.39}$$

Solving $\underline{\lambda}_1$ from Eqs. (5.37) and the boundary conditions (5.32) and (5.39), and then substituting that value in Eq. (5.38), $\underline{\omega}_B$ will be solved.

3. Solutions for $\underline{\lambda}_1$ and $\underline{\omega}_B(t)$

To solve for $\underline{\lambda}_1$, Eq. (5.1) is expressed as

$$\ddot{\underline{\omega}}_B + A\dot{\underline{\omega}}_B = f_1(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{w}, \underline{\theta}^*, t) \tag{5.40}$$

where

$$\underline{w} = \sum_i (\varepsilon')^i \underline{w}_i$$

The functions $\underline{w}_i$ are given by Eqs. (4.53) and (4.67). Then Eq. (5.37) reduces to

$$\ddot{\underline{\lambda}}_1 - (A - \frac{\partial \underline{f}_1}{\partial \dot{\underline{\omega}}_B})^T \dot{\underline{\lambda}}_1 - \left[\frac{\partial \underline{f}_1}{\partial \underline{\omega}_B} - \frac{d}{dt}\frac{\partial \underline{f}_1}{\partial \dot{\underline{\omega}}_B} + \frac{\partial \underline{f}_1}{\partial \underline{w}}\ \frac{\partial \bar{\underline{w}}}{\partial \underline{\omega}_B}\right]^T \underline{\lambda}_1 = 0 \tag{5.41}$$

Since $\bar{\underline{w}}(t)$ is a "bang-bang" control, we have

$$\frac{\partial \bar{\underline{w}}}{\partial \underline{\omega}_B} = 0 \quad \text{for} \quad 0 < t < T .$$

Now using Eq. (4.46a) as the explicit form of Eq. (5.40), Eq. (5.41) becomes

$$B_1^T \ddot{\underline{\lambda}}_1 - B_2^T \dot{\underline{\lambda}}_1 + B_3^T \underline{\lambda}_1 = \varepsilon' (\frac{\partial \underline{f}}{\partial \dot{\underline{\omega}}_B})^T \dot{\underline{\lambda}}_1 + \varepsilon' \left[\frac{d}{dt}(\frac{\partial \underline{f}}{\partial \dot{\underline{\omega}}_B}) - \frac{\partial \underline{f}}{\partial \underline{\omega}_B}\right]^T \underline{\lambda}_1 \tag{5.42}$$

From Eq. (4.46a), the form of Eq. (5.38) is obtained as

$$B_1\ddot{\underline{\omega}}_B + B_2\dot{\underline{\omega}}_B + B_3\underline{\omega}_B - B_4 \ \{\int_0^t B_5(\tau-t)[S_{B1}\dot{\underline{\omega}}_B + S_{B3}\underline{\omega}_B](\tau)d\tau\}$$

$$+ [B_4 \int_0^t B_5(\tau-t)\delta_B^* \ \bar{\underline{u}}_B(\tau)d\tau + B_6\dot{\bar{\underline{u}}}_B + B_7\bar{\underline{u}}_B - \delta_A^*\bar{\underline{u}}_A]$$

$$+ \varepsilon'\{B_4 \int_0^t B_5(\tau-t)\delta_B^*[U_B(\frac{\partial \underline{f}}{\partial \underline{u}_B} - \frac{d}{dt} \ \frac{\partial \underline{f}}{\partial \dot{\underline{u}}_B})^T \underline{\lambda}_1](\tau)d\tau$$

$$+ B_6 \frac{d}{dt} [U_B(\frac{\partial \underline{f}}{\partial \underline{u}_B} - \frac{d}{dt} \ \frac{\partial \underline{f}}{\partial \dot{\underline{u}}_B})^T \underline{\lambda}_1] + B_7[U_B(\frac{\partial \underline{f}}{\partial \underline{u}_B} - \frac{d}{dt} \frac{\partial \underline{f}}{\partial \dot{u}_B})^T \underline{\lambda}_1]$$

$$- \delta_A^* U_A(\frac{\partial \underline{f}}{\partial \underline{u}_A})^T \underline{\lambda}_1 + \underline{f}(\dot{\underline{\omega}}_B, \underline{\omega}_B, \bar{\underline{u}}_A, \bar{\underline{u}}_B)\} = - B_4 F_1 \underline{\theta}^* \tag{5.43}$$

Equations (5.42) and (5.43) are rewritten as

$$B_1^T \ddot{\underline{\lambda}}_1 - B_2^T \dot{\underline{\lambda}}_1 + B_3^T \underline{\lambda}_1 = \varepsilon' [g_1 \dot{\underline{\lambda}}_1 + g_2 \underline{\lambda}_1] \tag{5.44}$$

and

$$B_1 \ddot{\underline{\omega}}_B + B_2 \dot{\underline{\omega}}_B + B_3 \underline{\omega}_B - B_4 \{\int_0^t B_5(\tau - t)(S_{B1} \dot{\underline{\omega}}_B + S_{B3} \underline{\omega}_B)(\tau) d\tau\}$$

$$+ \bar{\underline{w}}(t) + B_4 F_1 \underline{\theta}^* = \varepsilon' [\int_0^t g_3(t,\tau) \underline{\lambda}_1(\tau) d\tau + g_4 \dot{\underline{\lambda}}_1 + g_5 \underline{\lambda}_1 - \underline{f}] \tag{5.45}$$

where

$$\bar{\underline{w}}(t) = B_4 \int_0^t B_5(\tau - t) \delta_B^* \bar{\underline{u}}_B(\tau) d\tau + B_6 \dot{\bar{\underline{u}}}_B(t) + B_7 \bar{\underline{u}}_B(t) - \delta_A^* \bar{\underline{u}}_A(t) \tag{5.46}$$

and $g_i(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B)$ are appropriately defined. Equations (5.44) and (5.45) will now be solved by the perturbation procedure sketched out in the deterministic case.

We begin the solution with the series

$$\underline{\omega}_B = \underline{x}_0' + \varepsilon' \underline{x}_1' + (\varepsilon')^2 \underline{x}_2' + (\varepsilon')^3 \underline{x}_3' + \ldots \tag{5.47}$$

$$\underline{\lambda}_1 = \underline{\lambda}_{10} + \varepsilon' \underline{\lambda}_{11} + (\varepsilon')^2 \underline{\lambda}_{12} + (\varepsilon')^3 \underline{\lambda}_{13} + \ldots \tag{5.48}$$

$$\bar{\underline{w}} = \bar{\underline{w}}_0 + \varepsilon' \bar{\underline{w}}_1 + (\varepsilon')^2 \bar{\underline{w}}_2 + (\varepsilon')^3 \bar{\underline{w}}_3 + \ldots \tag{5.49}$$

and Taylor's expansions of g_i and $\underline{f}$ about the point $\underline{v}_0'$, where $\underline{v}_i'$, $i = 0,1,2,\ldots$, are given by

$$\underline{v}_i' = [\dot{\underline{x}}_i', \underline{x}_i', \bar{\underline{u}}_{Ai}, \bar{\underline{u}}_{Bi}]^T . \tag{5.50}$$

Substituting these series in Eqs. (5.44) and (5.45), and separating the coefficients of $(\varepsilon')^0$, $(\varepsilon')^1$, $(\varepsilon')^2$, etc., we obtain the following equations:

$$B_1^T \ddot{\underline{\lambda}}_{10} - B_2^T \dot{\underline{\lambda}}_{10} + B_3^T \underline{\lambda}_{10} = 0 \tag{5.51}$$

$$B_1 \ddot{\underline{x}}_0' + B_2 \dot{\underline{x}}_0' + B_3 \underline{x}_0' - B_4 \{\int_0^t B_5(\tau - t)(S_{B1} \dot{\underline{x}}_0' + S_{B3} \underline{x}_0')(\tau) d\tau\}$$

$$+ \bar{\underline{w}}_0 + B_4 F_1 \underline{\theta}^* = 0 \tag{5.52}$$

$$B_1^T\ddot{\underline{\lambda}}_{11} - B_2^T\dot{\underline{\lambda}}_{11} + B_3^T\underline{\lambda}_{11} = g_1(v_0')\dot{\underline{\lambda}}_{10} + g_2(v_0')\underline{\lambda}_{10} \tag{5.53}$$

$$B_1\ddot{\underline{x}}_1' + B_2\dot{\underline{x}}_1' + B_3\underline{x}_1' - B_4\int_0^t B_5(\tau-t)[S_{B1}\dot{\underline{x}}_1' + S_{B3}\underline{x}_1'](\tau)d\tau$$

$$+ \bar{\underline{w}}_1 = \int_0^t [g_3(\underline{v}_0',t,\tau)\underline{\lambda}_{10}(\tau)]d\tau + g_4(\underline{v}_0')\dot{\underline{\lambda}}_{10} + g_5(\underline{v}_0')\underline{\lambda}_{10} - \underline{f}(\underline{v}_0') \tag{5.54}$$

$$B_1^T\ddot{\underline{\lambda}}_{12} - B_2^T\dot{\underline{\lambda}}_{12} + B_3^T\underline{\lambda}_{12} = g_1(\underline{v}_0')\dot{\underline{\lambda}}_{11} + g_2(\underline{v}_0')\underline{\lambda}_{11} + \underline{\nabla}g_1(\underline{v}_0')\underline{v}_1'\dot{\underline{\lambda}}_{10}$$

$$+ \underline{\nabla}g_2(\underline{v}_0')\underline{v}_1'\underline{\lambda}_{10} \tag{5.55}$$

$$B_1\ddot{\underline{x}}_2' + B_2\dot{\underline{x}}_2' + B_3\underline{x}_2' - B_4\int_0^t B_5(\tau-t)[S_{B1}\dot{\underline{x}}_2' + S_{B3}\underline{x}_2'](\tau)d\tau + \bar{\underline{w}}_2$$

$$= \int_0^t [g_3(\underline{v}_0',t,\tau)\underline{\lambda}_{11}(\tau) + \underline{\nabla}g_3(v_0',t,\tau)\underline{\lambda}_{10}(\tau)]d\tau + g_4(\underline{v}_0')\dot{\underline{\lambda}}_{11}$$

$$+ g_5(\underline{v}_0')\underline{\lambda}_{11} + \underline{\nabla}g_4(\underline{v}_0')\underline{v}_1'\dot{\underline{\lambda}}_{10} + \underline{\nabla}g_5(\underline{v}_0')\underline{v}_1'\underline{\lambda}_{10} - \underline{\nabla}\,\underline{f}(\underline{v}_0')\underline{v}_1' \tag{5.56}$$

Equations for higher powers of ε' can be obtained similarly. The homogeneous equations for Eqs. (5.51), (5.53) and (5.55) are given by

$$B_1^T\ddot{\underline{\lambda}}_{1i} - B_2^T\dot{\underline{\lambda}}_{1i} + B_3^T\underline{\lambda}_{1i} = 0 \tag{5.57}$$

Similarly, the homogeneous equations for $\underline{x}_i'$ are

$$B_1\ddot{\underline{x}}_i' + B_2\dot{\underline{x}}_i' + B_3\underline{x}_i' = 0 \tag{5.58}$$

The homogeneous system (5.58) is identical to that of Eq. (4.46a). So, as was done before, $\phi_1(t)$ and $\phi_2(t)$ are taken to be the fundamental matrices of Eq. (5.58). It is also seen that Eq. (5.52) is identical to Eq. (4.50). But $\underline{x}_0'$ differs from $\underline{x}_0$, as obtained in Eq. (4.54) due to the initial conditions.

Now, comparing Eqs. (5.51) and (5.58), we see that $\underline{\lambda}_{10}(t)$ is given by

$$\underline{\lambda}_{10}(t) = \phi_1(-t)\underline{a}_{10} + \phi_2(-t)\underline{a}_{20} \tag{5.59}$$

where $\underline{a}_{10}$ and $\underline{a}_{20}$ are constant vectors. Also, we use Eq. (4.54) to obtain the solution of Eq. (5.52) as

$$\underline{x}_0'(t) = \phi_1(t)\underline{b}_{10} + \phi_2(t)\underline{b}_{20} - \int_0^t \phi_2(t-\tau)B_4\{F_1(\tau)\underline{\theta}^*$$

$$- \int_0^\tau B_5(s-\tau)[S_{B1}\dot{\underline{x}}_0'(0) + S_{B3}\underline{x}_0'(0)]ds\}d\tau - \int_0^t \phi_2(t-\tau)\bar{\underline{w}}_0(\tau)d\tau \tag{5.60}$$

where $\underline{b}_{10}$ and $\underline{b}_{20}$ are constant vectors.

To solve Eqs. (5.53) and (5.54), we use the following identities:

$$\dot{\phi}_1(t) = -[\phi_2(t)]B_1^{-1}B_3 \tag{5.61}$$

$$\dot{\phi}_2(t) = B_1^{-1}B_3[\phi_1(t)]B_3^{-1}B_1 - B_1^{-1}B_2\phi_2(t) \tag{5.62}$$

Then from Eqs. (5.59), (5.61) and (5.62), Eq. (5.53) is expressed as

$$B_1^T\ddot{\underline{\lambda}}_{11} - B_2^T\dot{\underline{\lambda}}_{11} + B_3^T\underline{\lambda}_{11} = g_2(\underline{v}_0')\phi_1(-t)\underline{a}_{10} - g_1(\underline{v}_0')B_1^{-1}B_3\phi_1(-t)B_3^{-1}B_1\underline{a}_{20}$$

$$+ g_1(\underline{v}_0')\phi_2(-t)B_1^{-1}B_3\underline{a}_{10} + [g_1B_1^{-1}B_2 + g_2](\underline{v}_0')\phi_2(-t)\underline{a}_{20} \tag{5.63}$$

The formal solution of Eq. (5.63) is given by

$$\underline{\lambda}_{11}(t) = \phi_1(-t)\underline{a}_{11} + \phi_2(-t)\underline{a}_{21} + \int_0^t \phi_2(\tau-t)\{g_2(\underline{v}_0')\phi_1(-\tau)\underline{a}_{10}$$

$$- g_1(\underline{v}_0')B_1^{-1}B_3\phi_1(-\tau)B_3^{-1}B_1\underline{a}_{20} + g_1(v_0')\phi_2(-\tau)B_1^{-1}B_3\underline{a}_{10}$$

$$+ [g_1B_1^{-1}B_2 + g_2](\underline{v}_0')\phi_2(-\tau)\underline{a}_{20}\}d\tau \tag{5.64}$$

The integral in Eq. (5.64) will produce secular terms. To eliminate those, we apply the condition

$$\int_0^{T_1} \phi_2(t-T_1)\{g_2\phi_1(-t)\underline{a}_{10} - g_1B_1^{-1}B_3\phi_1(-t)B_3^{-1}B_1\underline{a}_{20} + g_1\phi_2(-t)B_1^{-1}B_3\underline{a}_{10}$$

$$+ [g_1B_1^{-1}B_2 + g_2]\phi_2(-t)\underline{a}_{20}\}dt = 0 \tag{5.65}$$

where T_1 is such that

$$\text{Det}[\int_0^{T_1} \phi_2(t-T_1)\phi_1(-t)dt] = 0 \tag{5.66}$$

Equation (5.65) is of the form

$$\underline{a}_{20} = K_0(\underline{b}_{10}, \underline{b}_{20})\underline{a}_{10} \tag{5.67}$$

So from Eq. (5.59) the true solution for $\underline{\lambda}_{10}$ is given by

$$\underline{\lambda}_{10}(t) = [\phi_1(-t) + \phi_2(-t)k_0]\underline{a}_{10} \tag{5.68}$$

From Eq. (5.60), we obtain

$$\underline{x}_0'(0) = \phi_1(0)\underline{b}_{10} \tag{5.69}$$

$$\dot{\underline{x}}_0'(0) = \dot{\phi}_2(0)b_{20} \tag{5.70}$$

and thus

$$\underline{x}_0' = \underline{x}_0'(\phi_1, \phi_2, \underline{b}_{10}, \underline{b}_{20}, \underline{\theta}^*)$$

Now the formal solution of Eq. (5.54) is given by

$$\underline{x}_1'(t) = \phi_1(t)\underline{b}_{11} + \phi_2(t)\underline{b}_{21} + \int_0^t \phi_2(t-\tau)\{\int_0^\tau g_3(\underline{b}_{10},\underline{b}_{20},\tau,s)\underline{\lambda}_{10}(\underline{a}_{10},s)ds$$

$$+ B_4 \int_0^\tau B_5(s-\tau)[S_{B1}\underline{x}_1'(0) + S_{B3}\underline{x}_1'(0)]ds + g_4(\underline{b}_{10},\underline{b}_{20},\tau)\dot{\underline{\lambda}}_{10}(\underline{a}_{10},\tau)$$

$$+ g_5(\underline{b}_{10},\underline{b}_{20},\tau)\underline{\lambda}_{10}(\underline{a}_{10},\tau) - f(\underline{b}_{10},\underline{b}_{20},\tau) - \bar{\underline{w}}_1(\underline{b}_{10},\underline{b}_{20},\tau)\}d\tau \tag{5.71}$$

To eliminate the secular terms from Eq. (5.71), we impose the condition

$$\int_0^{T_1} \phi_2(T_1-t)\{\int_0^t [g_3(\underline{b}_{10},\underline{b}_{20},t,\tau)\underline{\lambda}_{10}(\underline{a}_{10},\tau)]d\tau + g_4(\underline{b}_{10},\underline{b}_{20},t)\dot{\underline{\lambda}}_{10}(\underline{a}_{10},t)$$

$$+ g_5(\underline{b}_{10},\underline{b}_{20},t)\underline{\lambda}_{10}(\underline{a}_{10},t) - f(\underline{b}_{10},\underline{b}_{20},t) - \bar{\underline{w}}_1(\underline{b}_{10},\underline{b}_{20},t)\}dt = 0 \tag{5.72}$$

where T_1 is given by Eq. (5.66). Equation (5.72) is of the form

$$\underline{b}_{20} = K_0'(\underline{a}_{10},\underline{b}_{10}) \tag{5.73}$$

Substituting the value of $\underline{b}_{20}$ from Eq. (5.73) into Eq. (5.60), the solution of $\underline{\dot{x}}_0'(t)$ is obtained as

$$\underline{x}_0'(t) = \underline{x}_0'[\phi_1(t), \phi_2(t), \underline{a}_{10}, \underline{b}_{10}, \underline{\theta}^*, t] \tag{5.74}$$

Proceeding in this way with Eqs. (5.55) and (5.56), we obtain

$$\underline{a}_{21} = k_1(\underline{b}_{10}, \underline{b}_{11})\underline{a}_{11} \tag{5.75}$$

$$\underline{b}_{21} = k_1'(\underline{a}_{10}, \underline{a}_{11}, \underline{b}_{10}, \underline{b}_{11}) \tag{5.76}$$

In solving Eqs. (5.55) and (5.56), we assume zero initial conditions for $\underline{\lambda}_{12}$, $\underline{\dot{\lambda}}_{12}$, $\underline{x}_2'$ and $\underline{\dot{x}}_2'$, so that no new variables are introduced.

This iteration procedure is stopped after obtaining three term approximations for $\underline{\lambda}_1$ and $\underline{\omega}_B$. The constants a_{10} and a_{11} are obtained from the following boundary conditions given by Eqs. (5.39) and (5.6):

$$\underline{\lambda}_{10}(0) = R_0^{-1}(B_2^{-1})^T B_1^T[\underline{v}_1(0) + \underline{v}_2(\underline{q}(0), 0)] \tag{5.77}$$

$$\underline{\lambda}_{11}(0) = 0 = \underline{a}_{11} \tag{5.78}$$

The time interval (0,T) for the optimal estimates to be valid is obtained from Eq. (5.32) as

$$\underline{\dot{\lambda}}_{10}(T) = B_2^T(B_1^{-1})^T\underline{\lambda}_{10}(T) \tag{5.79}$$

We also set

$$\underline{b}_{21} = 0 \tag{5.80}$$

so as to obtain $\underline{b}_{11}$ as a function of the known constants $\underline{a}_{10}$ and $\underline{b}_{10}$ from Eq. (5.76).

Combining the results of the preceding analysis, we obtain the most likely estimates of

$$\underline{\omega}_B = \underline{\omega}_B(\underline{a}_{10}, \underline{b}_{10}, R_0, R_1, U_A, U_B, P_0, t) \tag{5.81}$$

Setting

$$a_{10} = R_0 = R_1 = U_A = U_B = P_0 = 0$$

we immediately obtain the solution of Eq. (4.46) giving the mean value $\bar{\underline{\omega}}_B$ of $\underline{\omega}_B$ as

$$\bar{\underline{\omega}}_B = \underline{\omega}_B(0, \underline{b}_{10}, 0, 0, 0, 0, 0, t) \tag{5.82}$$

Let Q be defined as

$$Q(t) = E\{[\underline{\omega}_B - \bar{\underline{\omega}}_B][\underline{\omega}_B - \bar{\underline{\omega}}_B]^T\} \tag{5.83}$$

The function Q will be calculated from the known expressions (5.81) and (5.82).

4. <u>Deterministic and Random Solutions for</u> $\underline{q}$

a) Particular solutions

We now consider Eqs. (2.112) and (2.113) which are repeated here without the subscripts A and B, as

$$A_1\ddot{\underline{q}} + A_2(\underline{\omega},t)\dot{\underline{q}} + A_3(\underline{\omega},t)\underline{q} = \underline{A}_4(\underline{\omega},t) \tag{5.84}$$

Particular solutions for $\underline{q}$ are given in Eq. (2.116b) which is

$$\underline{q}_p = M_4^{-1}(\underline{\omega},t)\underline{M}_5(\underline{\omega},t) \tag{5.85}$$

where $\underline{M}_4$ and $\underline{M}_5$ are defined in Eq. (2.116). Using the deterministic and random solutions of $\underline{\omega}_B$, given in Eqs. (5.82) and (5.83), the corresponding deterministic and random solutions for $\underline{q}_p$ are obtained from Eq. (5.85).

b) Homogeneous solutions

The homogeneous solutions for $\underline{q}$ are the solutions of

$$A_1\ddot{\underline{q}} + A_2(\underline{\omega},t)\dot{\underline{q}} + A_3(\underline{\omega},t)\underline{q} = 0 . \tag{5.86}$$

This linear equation with time-dependent coefficients could be solved analytically if A_2 and A_3 were periodic. But, in general, the roots of the characteristic equation of Eq. (5.58) will be of the form $(\sigma_j \pm i\sigma_j')$, $j = 1,2,3$. This makes $\underline{\omega}_B$, and hence A_2 and A_3, nonperiodic. So, Eq. (5.86) is solved as follows:

The coefficient A_2 is expressed as

$$A_2(\underline{\omega},t) = A_{20} + A_{211} + A_{212} + A_{213} + A_{221} + A_{222} + A_{223} + A_{224} + A_{225} + A_{226} + \cdots \tag{5.87}$$

where

$$A_{211} = e^{\sigma_1 t} A'_{211}; \quad A_{212} = e^{\sigma_2 t} A'_{212}; \quad A_{213} = e^{\sigma_3 t} A'_{213};$$

$$A_{221} = e^{2\sigma_1 t} A'_{221}; \quad A_{222} = e^{2\sigma_2 t} A'_{222}; \quad A_{223} = e^{2\sigma_3 t} A'_{223};$$

$$A_{224} = e^{(\sigma_1+\sigma_2)t} A'_{224}; \quad A_{225} = e^{(\sigma_2+\sigma_3)t} A'_{225}; \quad A_{226} = e^{(\sigma_3+\sigma_1)t} A'_{226};$$

etc.

Similarly, $A_3(\underline{\omega},t)$ is expressed as

$$A_3(\underline{\omega},t) = A_{30} + A_{311} + A_{312} + A_{313} + \cdots \tag{5.88}$$

where A_{3ij} are defined analogously. A'_{kij} are periodic functions of σ'_i, $i = 1,2,3$. A_{20} and A_{30} are constants. Let $\underline{q}$ be expressed as the sum

$$\underline{q} = \underline{q}_0 + \underline{q}_{11} + \underline{q}_{12} + \underline{q}_{13} + \underline{q}_{21} + \underline{q}_{22} + \underline{q}_{23} + \underline{q}_{24} + \underline{q}_{25} + \underline{q}_{26} + \cdots \tag{5.89}$$

where

$$\underline{q}_{11} = e^{\sigma_1 t} \underline{q}'_{11}; \quad \underline{q}_{12} = e^{\sigma_2 t} \underline{q}'_{12}; \quad \underline{q}_{13} = e^{\sigma_3 t} \underline{q}'_{13};$$

$$\underline{q}_{21} = e^{2\sigma_1 t} \underline{q}'_{21}; \quad \underline{q}_{22} = e^{2\sigma_2 t} \underline{q}'_{22}; \quad \underline{q}_{23} = e^{2\sigma_3 t} \underline{q}'_{23};$$

$$\underline{q}_{24} = e^{(\sigma_1+\sigma_2)t} \underline{q}'_{24}; \quad \underline{q}_{25} = e^{(\sigma_2+\sigma_3)t} \underline{q}'_{25}; \quad \underline{q}_{26} = e^{(\sigma_3+\sigma_1)t} \underline{q}'_{26};$$

etc.

The expressions for A_2, A_3 and $\underline{q}$ given by Eqs. (5.87), (5.88) and (5.89) are now substituted into Eq. (5.86). From the resulting equation, the coefficients of $e^{\sigma_1 t}$, $e^{\sigma_2 t}$, etc. are separated and set equal to zero. This procedure gives the following equations:

$$A_1\ddot{\underline{q}}_0 + A_{20}\dot{\underline{q}}_0 + A_{30}\underline{q}_0 = 0 \tag{5.90}$$

$$A_1\ddot{\underline{q}}_{11} + A_{20}\dot{\underline{q}}_{11} + A_{30}\underline{q}_{11} = - A_{211}\dot{\underline{q}}_0 - A_{311}\underline{q}_0 \tag{5.91}$$

$$A_1\ddot{\underline{q}}_{12} + A_{20}\dot{\underline{q}}_{12} + A_{30}\underline{q}_{12} = - A_{212}\dot{\underline{q}}_0 - A_{312}\underline{q}_0 \tag{5.92}$$

$$A_1\ddot{\underline{q}}_{13} + A_{20}\dot{\underline{q}}_{13} + A_{30}\underline{q}_{13} = - A_{213}\dot{\underline{q}}_0 - A_{313}\underline{q}_0 \tag{5.93}$$

$$A_1\ddot{\underline{q}}_{21} + A_{20}\dot{\underline{q}}_{21} + A_{30}\underline{q}_{21} = - A_{221}\dot{\underline{q}}_0 - A_{321}\underline{q}_0 - A_{211}\dot{\underline{q}}_{11} - A_{311}\underline{q}_{11} \tag{5.94}$$

$$A_1\ddot{\underline{q}}_{22} + A_{20}\dot{\underline{q}}_{22} + A_{30}\underline{q}_{22} = - A_{222}\dot{\underline{q}}_0 - A_{322}\underline{q}_0 - A_{212}\dot{\underline{q}}_{12} - A_{312}\underline{q}_{12} \tag{5.95}$$

$$A_1\ddot{\underline{q}}_{23} + A_{20}\dot{\underline{q}}_{23} + A_{30}\underline{q}_{23} = - A_{223}\dot{\underline{q}}_0 - A_{323}\underline{q}_0 - A_{213}\dot{\underline{q}}_{13} - A_{313}\underline{q}_{13} \tag{5.96}$$

$$A_1\ddot{\underline{q}}_{24} + A_{20}\dot{\underline{q}}_{24} + A_{30}\underline{q}_{24} = - A_{224}\dot{\underline{q}}_0 - A_{324}\underline{q}_0 - A_{211}\dot{\underline{q}}_{12} - A_{311}\underline{q}_{12} - A_{212}\dot{\underline{q}}_{11} - A_{312}\underline{q}_{11} \tag{5.97}$$

$$A_1\ddot{\underline{q}}_{25} + A_{20}\dot{\underline{q}}_{25} + A_{30}\underline{q}_{25} = - A_{225}\dot{\underline{q}}_0 - A_{325}\underline{q}_0 - A_{212}\dot{\underline{q}}_{13} - A_{312}\underline{q}_{13} - A_{213}\dot{\underline{q}}_{12} - A_{313}\underline{q}_{12} \tag{5.98}$$

$$A_1\ddot{\underline{q}}_{26} + A_{20}\dot{\underline{q}}_{26} + A_{30}\underline{q}_{26} = - A_{226}\dot{\underline{q}}_0 - A_{326}\underline{q}_0 - A_{211}\dot{\underline{q}}_{13} - A_{311}\underline{q}_{13} - A_{213}\dot{\underline{q}}_{13} - A_{313}\underline{q}_{11} \tag{5.99}$$

Equations corresponding to the cubic and higher order terms in e^{σ_i} are neglected. It is seen that $\underline{q}_0$ is the basic solution which governs all the other terms. Let ψ_1 and ψ_2 be the fundamental matrices of Eq. (5.90). Then $\underline{q}_0$ is given by

$$\underline{q}_0 = \psi_1 \underline{c}_{10} + \psi_2 \underline{c}_{20} \tag{5.100}$$

where $\underline{c}_{10}$ and $\underline{c}_{20}$ are constants to be obtained from initial conditions and the bifurcation equations. The initial conditions of $\underline{q}$ are lumped on $\underline{q}_0$, so that all the $\underline{q}_{ij}$ in Eq. (5.89) have zero initial conditions. Now we define

$$A_{21} = A_{211} + A_{212} + A_{213} \tag{5.101}$$

and

$$A_{31} = A_{311} + A_{312} + A_{313} \tag{5.102}$$

so that A_{21} and A_{31} are the components of A_2 and A_3 which are linear in $\underline{\omega}_B$ and $\dot{\underline{\omega}}_B$. Then, adding together Eqs. (5.91), (5.92) and (5.93), we get

$$\underline{q}_1 = \underline{q}_{11} + \underline{q}_{12} + \underline{q}_{13} \doteq - \int_0^t \psi_2(t-\tau)[A_{21}\dot{\underline{q}}_0 + A_{31}\underline{q}_0](\tau)d\tau \tag{5.103}$$

Hence

$$\dot{\underline{q}}_1(t) = - \int_0^t \dot{\psi}_2(t-\tau)[A_{21}\dot{\underline{q}}_0 + A_{31}\underline{q}_0](\tau)d\tau \tag{5.104}$$

Now, adding together Eqs. (5.91) through (5.99), we get

$$\begin{aligned}\underline{q} - \underline{q}_0 = - \int_0^t \psi_2(t-\tau)\{&(A_2-A_{20})\dot{\underline{q}}_0 + (A_3-A_{30})\underline{q}_0 \\ &- A_{21} \int_0^\tau \dot{\psi}_2(\tau-s)[A_{21}\dot{\underline{q}}_0 + A_{31}\underline{q}_0](s)ds \\ &- A_{31} \int_0^\tau \psi_2(\tau-s)[A_{21}\dot{\underline{q}}_0 + A_{31}\underline{q}_0](s)ds\}(\tau)d\tau\end{aligned} \tag{5.105}$$

The above equation gives an explicit solution for $\underline{q}$ as the functions $A_2(\underline{\omega},t)$ and $A_3(\underline{\omega},t)$ are known from the preceding analysis. It is also evident that the integrals in Eq. (5.105) generate secular terms which must be eliminated by imposing a relation between the constants $\underline{c}_{10}$ and $\underline{c}_{20}$ introduced in Eq. (5.100). To do this, we proceed as follows:

Differentiating Eq. (5.100), we get

$$\dot{\underline{q}}_0(t) = \dot{\psi}_1(t)\underline{c}_{10} + \dot{\psi}_2(t)\underline{c}_{20} \tag{5.106}$$

Substituting the values of $\underline{q}_0$ and $\dot{\underline{q}}_0$ from Eqs. (5.100) and (5.106) into the integrals in Eq. (5.105), we get

$$\begin{aligned}
\underline{q} = \underline{q}_0 &- \{\int_0^t \psi_2(t-\tau)[(A_2-A_{20})\dot{\psi}_1 + (A_3-A_{30})\psi_1](\tau)d\tau\}\underline{c}_{10} \\
&- \{\int_0^t \psi_2(t-\tau)[(A_2-A_{20})\dot{\psi}_2 + (A_3-A_{30})\psi_2](\tau)d\tau\}\underline{c}_{20} \\
&+ \{\int_0^t \psi_2(t-\tau)[A_{21} \int_0^\tau \dot{\psi}_2(\tau-s)(A_{21}\dot{\psi}_1 + A_{31}\psi_1)(s)ds \\
&\quad + A_{31} \int_0^\tau \psi_2(\tau-s)(A_{21}\dot{\psi}_1 + A_{31}\psi_1)(s)ds](\tau)d\tau\}\underline{c}_{10} \\
&+ \{\int_0^t \psi_2(t-\tau)[A_{21} \int_0^\tau \dot{\psi}_2(\tau-s)(A_{21}\dot{\psi}_2 + A_{31}\psi_2)(s)ds \\
&\quad + A_{31} \int_0^\tau \psi_2(\tau-s)(A_{21}\dot{\psi}_2 + A_{31}\psi_2)(s)ds](\tau)d\tau\}\underline{c}_{20}
\end{aligned} \tag{5.107}$$

Let $\psi_2(-t)$ and $\psi_1(t)$ be orthogonal in an interval $[0,T_2]$ such that

$$\text{Det}[\int_0^{T_2} \psi_2(T_2 - t)\psi_1(t)dt] = 0 \tag{5.108}$$

Then the condition for the secular terms to be absent is given by

$$\begin{aligned}
&\{ \int_0^{T_2} \psi_2(T_2-t)[(A_2-A_{20})\dot{\psi}_1 + (A_3-A_{30})\psi_1 - A_{21} \int_0^t \dot{\psi}_2(t-\tau)(A_{21}\dot{\psi}_1 + A_{31}\psi_1)(\tau)d\tau \\
&\quad - A_{31} \int_0^t \psi_2(t-\tau)(A_{21}\dot{\psi}_1 + A_{31}\psi_1)(\tau)d\tau]dt\}\underline{c}_{10} \\
&+ \{ \int_0^{T_2} \psi_2(T_2-t)[(A_2-A_{20})\dot{\psi}_2 + (A_3-A_{30})\psi_2 - A_{21} \int_0^t \dot{\psi}_2(t-\tau)(A_{21}\dot{\psi}_2 + A_{31}\psi_2)(\tau)d\tau \\
&\quad - A_{31} \int_0^t \psi_2(t-\tau)(A_{21}\dot{\psi}_2 + A_{31}\psi_1)(\tau)d\tau]dt\}\underline{c}_{20} = 0
\end{aligned} \tag{5.109}$$

Equation (5.109) can be expressed as

$$L_1\underline{c}_{10} + L_2\underline{c}_{20} = 0 \tag{5.110}$$

Hence, from Eqs. (5.100) and (5.110), we obtain

$$\underline{q}_0 = [\psi_1 - \psi_2 L_2^{-1} L_1]\underline{c}_{10} \tag{5.111}$$

From the above analysis, we see that $\underline{q}(t)$ coincides with $\underline{q}_0(t)$ for $t = 0, T_2, 2T_2$, etc.

The complete solution is obtained by adding the particular solution and the homogeneous solutions given by Eqs. (5.85) and (5.107). But it may be computationally unfeasible to calculate the inverse of the functional matrix $M_4(\underline{\omega},t)$. So we now present an alternative method of obtaining the complete solution for $\underline{q}(t)$.

c) Alternative method for the complete solution

In this method, Eq. (5.84) is considered. The vector $\underline{A}_4$ is expressed as

$$\underline{A}_4(\underline{\omega},t) = \underline{A}_{40} + \underline{A}_{411} + \underline{A}_{412} + \underline{A}_{413} + \underline{A}_{421} + \underline{A}_{422} + \underline{A}_{423} + \underline{A}_{424} + A_{425} + A_{426} \tag{5.112}$$

where

$$\underline{A}_{411} = e^{\sigma_1 t}\underline{A}'_{411}; \quad \underline{A}_{412} = e^{\sigma_2 t}\underline{A}'_{412}; \quad \underline{A}_{413} = e^{\sigma_3 t}\underline{A}'_{413};$$

$$\underline{A}_{421} = e^{2\sigma_1 t}A'_{421}; \quad \underline{A}_{422} = e^{2\sigma_2 t}\underline{A}'_{422}; \quad \underline{A}_{423} = e^{2\sigma_3 t}\underline{A}'_{423};$$

$$\underline{A}_{424} = e^{(\sigma_1+\sigma_2)t}A'_{424}; \quad \underline{A}_{425} = e^{(\sigma_2+\sigma_3)t}\underline{A}'_{425}; \quad \underline{A}_{426} = e^{(\sigma_3+\sigma_1)t}\underline{A}'_{426}.$$

In the above $\underline{A}'_{4ij}$ are periodic functions.

Using Eqs. (5.87), (5.88), (5.89) and (5.112) the equations generated by the coefficients of the exponential functions from Eq. (5.84) are

$$A_1\ddot{q}_0 + A_{20}\dot{q}_0 + A_{30}q_0 = A_{40} \quad (5.113)$$

$$A_1\ddot{q}_{1j} + A_{20}\dot{q}_{1j} + A_{30}q_{1j} = A_{41j} - A_{21j}\dot{q}_0 - A_{31j}q_0 \quad (5.114)$$

where $j = 1,2,3.$

$$A_1\ddot{q}_{2k} + A_{20}\dot{q}_{2k} + A_{30}q_{2k} = A_{42k} - A_{22k}\dot{q}_0 - A_{32k}q_0 - A_{21k}\dot{q}_{1k} - A_{31k}q_{1k}$$

for $k = 1,2,3.$ (5.115)

Equations corresponding to Eq. (5.97), (5.98) and (5.99) are similarly obtained. From Eq. (5.113),

$$\underline{q}_0 = \psi_1(t)\underline{c}_{10} + \psi_2(t)\underline{c}_{20} + \int_0^t \psi_2(t-\tau)\underline{A}_{40}d\tau \quad (5.116)$$

Defining

$$A_{41} = A_{411} + A_{412} + A_{413} \quad (5.117)$$

we get

$$\underline{q}_1 = \int_0^t \psi_2(t-\tau)[A_{41} - A_{21}\dot{q}_0 - A_{31}q_0](\tau)d\tau \quad (5.118)$$

which corresponds to Eq. (5.103).

Proceeding as shown before, we get

$$\underline{q} = \underline{q}_0 + \int_0^t \psi_2(t-\tau)\{(\underline{A}_4-\underline{A}_{40}) - (A_2-A_{20})\dot{\underline{q}}_0 - (A_3-A_{30})\underline{q}_0$$

$$+ A_{21}\int_0^\tau \dot{\psi}_2(\tau-s)[A_{21}\dot{\underline{q}}_0 + A_{31}\underline{q}_0](s)ds$$

$$+ A_{31}\int_0^\tau \psi_2(\tau-s)[A_{21}\dot{\underline{q}}_0 + A_{31}\underline{q}_0](s)ds\}(\tau)d\tau \quad (5.119)$$

The bifurcation equation corresponding to Eq. (5.109) is then given by

$$L_1\underline{c}_{10} + L_2\underline{c}_{20} = \underline{L}_3 \quad (5.120)$$

where

$$\underline{L}_3 = \int_0^{T_2} \psi_2(T_2-\tau)\{(\underline{A}_4-\underline{A}_{40}) - (A_2-A_{20})\int_0^\tau \dot{\psi}_2(\tau-s)\underline{A}_{40}ds$$

$$- (A_3-A_{30})\int_0^\tau \psi_2(\tau-s)\underline{A}_{40}ds + A_{21}\int_0^\tau \dot{\psi}_2(\tau-s)[A_{21}(s)\int_0^s \dot{\psi}_2(s-p)\underline{A}_{40}dp +$$

$$+ A_{31}(s) \int_0^s \psi_2(s-p)\underline{A}_{40}dp]ds + A_{31} \int_0^\tau \psi_2(\tau-s)[A_{21}(s) \int_0^s \dot{\psi}_2(s-p)\underline{A}_{40}dp$$

$$+ A_{31}(s)\int_0^s \psi_2(s-p)\underline{A}_{40}dp]ds\}(\tau)d\tau \qquad (5.121)$$

Hence, from Eqs. (5.116) and (5.120), we get

$$\underline{q}_0(t) = [\psi_1 - \psi_2 L_2^{-1} L_1]\underline{c}_{10} + \psi_2 L_2^{-1} \underline{L}_3 + \int_0^t \psi_2(t-\tau)\underline{A}_{40}d\tau \qquad (5.122)$$

Considering zero initial conditions, Eq. (5.122) becomes

$$\underline{q}_0(\underline{\omega}_B,t) = \psi_2(t)L_2^{-1}L_3(\underline{\omega}_B) + \int_0^t \psi_2(t-\tau)\underline{A}_{40}d\tau \qquad (5.123)$$

Expressing $\underline{\omega}_B$ as given by Eq. (5.81) in Eqs. (5.123) and (5.119), we obtain the most likely estimates of

$$\underline{q}(t) = \underline{q}(\underline{a}_{10}, \underline{b}_{10}, R_0, R_1, U_A, U_B, \underline{P}_0, t) \qquad (5.124)$$

Using the expression (5.82) in Eqs. (5.123) and (5.119), we obtain the mean values of $\underline{q}(t)$, given by

$$\bar{\underline{q}}(t) = \bar{\underline{q}}(0, \underline{b}_{10}, 0, 0, 0, 0, 0, t) \qquad (5.125)$$

Equations (5.124) and (5.125) will now be used to obtain more accurate estimates of P_0 given by Eq. (5.5).

5. Pointing Error Analysis

a) Error for the body B

Let it be assumed now that the nominal angular velocities of the body B are zero. Let ψ_{Bi} be the absolute angular position vector of the i^{th} Earth Viewing Module of the body B. Let $\underline{v}_{3i}(t)$ be the intrinsic error of the i^{th} EVM. Let $\underline{v}_{4i}(t)$ be the pointing error of the i^{th} EVM with respect to the body B due to the flexibilities. The error $\underline{v}_{4i}(t)$ can be expressed as

$$\underline{v}_{4i}(t) = c_i^{**}\underline{q}(t) \quad (5.126)$$

where c_i^{**} are constant matrices for all i. Then, the most likely pointing error for the i^{th} EVM in the body B is given by

$$\underline{\psi}_{Bi} = \int_0^t \underline{\omega}_B(\underline{a}_{10}, \underline{b}_{10}, R_0, R_1, U_A, U_B, P_0, t)dt$$

$$+ c_i^{**}\underline{q}(\underline{a}_{10}, \underline{b}_{10}, R_0, R_1, U_A, U_B, P_0, t) + \underline{v}_{3i} \quad (5.127)$$

The mean error is then given by

$$\overline{\psi}_{B\,i} = \int_0^t \overline{\underline{\omega}}_B dt + c_i^{**}\overline{\underline{q}}(t) \quad (5.128)$$

where it is assumed that $\underline{v}_{3i}$ has zero mean. Let P_{Bi} be defined by

$$P_{Bi} = E\{[\underline{\psi}_{Bi}(t) - \overline{\underline{\psi}}_{Bi}(t)][\psi_{Bi}(t) - \overline{\underline{\psi}}_{Bi}(t)]^T\} \quad (5.129)$$

Then $P_{Bi}(t)$ will be calculated from Eqs. (5.127) and (5.128) when the statistical properties of $\underline{v}_{3i}$ are given. Thus Eqs. (5.127), (5.128) and (5.129) completely specify the error $\underline{\psi}_{Bi}$.

b) Error for the body A

Let $\underline{\psi}_{Ai}$ be the absolute angular position vector of the i^{th} EVM of the body A. Let $\underline{v}_{3i}(t)$ and $\underline{v}_{4i}(t)$ be the intrinsic error and the flexibility error of the i^{th} EVM of the body A. Then

$$\underline{\psi}_{Ai} = \int_0^t \underline{\omega}_B dt + \underline{\theta}(t) + \underline{v}_{4i}(t) + c_i^{**}\underline{q}(t) \quad (5.130)$$

The above equation gives the most likely values of $\underline{\psi}_{Ai}$. The mean value is given by

$$\overline{\underline{\psi}}_{Ai} = \int_0^t \overline{\underline{\omega}}_B dt + \overline{\underline{\theta}}(t) + c_i^{**}\overline{\underline{q}}(t) \quad (5.131)$$

where it is assumed that $\underline{v}_{4i}(t)$ has zero mean. Hence the variances P_{Ai} will be calculated from

$$P_{Ai} = E\{[\underline{\psi}_{Ai}(t) - \overline{\underline{\psi}}_{Ai}(t)][\underline{\psi}_{Ai}(t) - \overline{\underline{\psi}}_{Ai}(t)]^T\} \quad (5.132)$$

This completes the random pointing error analysis for the asymptotic equations of motion.

6. References

1. Saaty, T. L., and Bram, J., *Nonlinear Mathematics,* McGraw-Hill, 1964.

2. Saaty, T. L., *Modern Nonlinear Equations,* McGraw-Hill, 1967.