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Preface

This is the third quarterly report on Contract NAS5-21798, "Studies of Sounding and Imaging Measurements from Geostationary Satellites." Significant progress has been made on most tasks during the reporting period. The prototype McIDAS equipment has continued to work well.

Introduction

This third quarterly report covers work performed from 1 February 1973 through 30 April 1973. In summary, progress by task is as follows:

Task A. Investigations of Meteorological Data Processing Techniques

Substantial progress has been made towards correcting ATS line start errors. This includes the development of an algorithm to detect the right hand earth limb in very noisy ATS images and the development of a processing technique which uses the limb detection algorithms to correct raw ATS data. Line start errors have been corrected on 23 ATS images, thus allowing relatively accurate wind determination from otherwise unusable data.

Task B. Sun Glitter

Mr. Jack Kornfield, the student who has almost completed a Ph. D. thesis on the "Determination of Sea Surface Wind and Stress from Sunglint Observed by a Geostationary Satellite," is no longer at the University. However, he is still working to complete the thesis. In order to meet the University thesis requirements, it must be completed by October. Therefore, it will be available as a final report at that time--whether completed as a thesis or not.

Task D. Cloud Growth Rate

In a master's thesis now completed, the effects of changing sun-satellite-cloud geometry on ATS-3 viewed cloud brightness have been documented. The brightness contrast analyses within clouds were found to depend on scattering angles. For a large cloud mass different parts of the same clouds appeared to have different scattering properties; hence, different normalization corrections may be required. This suggests an important deficiency of present normalization procedures, which apply the same correction factor for all parts of the cloud.

Also considered were observations of the scattering phase function and the effects of cloud breaks smaller than camera resolution. Work is underway on the development of a normalization procedure which will use a theoretical multiple scattering program to generate data on the brightness of clouds with various thicknesses for varying sun-cloud-satellite geometries.

Task E. Comparative Studies in Satellite Stability

Appendices 6 and 7 and Part VI to Das' technical report have been completed. Part VI of the report is attached at the end of this summary as an appendix. Complete coding of the model is proceeding on schedule.

Task F. High Resolution Optics Study

The seminar on mesoscale monitoring and prediction has been completed. Students have studied the causes and behavior of tornadoes, hurricanes, heavy snow swaths, thunderstorms, hail, sleet, flash floods, clear air turbulence, fog, and lake and sea breezes. Key observables in the motion, thermal, and moisture fields were identified and means of communicating such information to the local Madison community explored. Two days of briefings were held to transmit this information from the students to SSEC physicists and engineers who will complete the study by investigating how the key observables can be measured; and, if time permits, by broadening the scope of the seminar subjects. We have reached two main conclusions so far: (1) SEOS will combine attributes necessary for mesoscale study which are not found in any other observing system, and (2) a high resolution vertical temperature and moisture sounding capability (in addition to imaging) is required to adequately treat mesoscale monitoring and prediction.

Operation of the McIDAS system has been virtually trouble-free. Use of the system has been so intensive, the two primary problems over the last few weeks have been scheduling users and near saturation of the disk storage capacity. Certain program objectives and emphases have been shifted to reflect the demands of the new Task F, High Resolution Optics Study.

Task Progress

<u>Task A.</u> <u>Investigations of Meteorological Data Processing Techniques</u>

During the past quarter, effort on this task has resulted in significant progress in the correction of ATS line start errors. Specific advances include:

1. Development of an algorithm to detect the right hand earth limb in very noisy ATS images;
2. Development of a processing technique which uses the limb detection algorithms to correct raw ATS data; and
3. Correction of line start errors in 23 ATS images, thus allowing relatively accurate wind determinations from otherwise unuseable data.

Each of these items is explained in the following series of discussions.

Development of Right Hand Algorithm

Development of a noise compensated threshold detection algorithm for locating an effective left hand earth edge has already been reported on in previous quarterly reports. The difference between the left hand and right hand edges of (some) ATS pictures is the amount of communication noise present. This noise on the right hand side is so large (for days 203, 204, of 1969) that an entirely new algorithm was needed.

Line plots of ATS data (day 204, 1969) near the right hand limb revealed large and frequent noise spikes both within the earth's disk and in space. Application of a simple filter (10 sample running mean) eliminated the spikes but did not remove lower frequency noise components. However, further filtering was not practical because the rise of the limb radiance profile would also be filtered out. Instead, the problem of falsely triggering on a low frequency noise burst was largely avoided by requiring the threshold to be exceeded at several points over a wide sample spacing simultaneously. In other words, the edge position was found at sample N if N was the largest sample number for which the ATS signal exceeded the specified threshold at N, N-1, N-2, N-5, N-10, N-20, and N-30. (A better, though more complicated procedure, would be to find the best fit position of a theoretical profile to the data.) As in the case of the left hand algorithm, the threshold is augmented by the mean space noise before testing against the data.

Test runs with thresholds of 5, 10, 20, 30, and 40 were made to find the threshold which yielded the smallest line to line variance. The 30 threshold was found to be the most stable. The lower thresholds appeared to be quite sensitive to noise; and the higher threshold appeared to respond to cloud cover variations. Unfortunately, it was not possible to check the threshold detection results against landmark measurements (as was done with the left hand algorithm) since no suitable landmarks were illuminated at the same time the right hand limb was illuminated. The only further verification available at this time is in the consistency of wind fields derived from data corrected from the algorithm results.

Development of a Data Processing Technique

Given a measured position of the earth edge for each line of each ATS image in a sequence, there are a number of ways in which that information can be used to improve the accuracy of cloud motion vectors. The method we have chosen after some experimentation is one which maintains maximum compatibility with already existing data processing software. The basic operations involved can be summarized as follows:

a) image navigation,

b) calculation of navigation predictions of earth edge position as a function of line number for each image,

c) threshold detection of earth edge position as a function of line number for each image, and

d) write new digital tapes which are identical to raw tapes except for line shifts equal to the differences between results of b) and c).

After d) is completed, images can be displayed and processed for wind sets using standard techniques.

The first step (image navigation), although a standard procedure in ATS data processing, required special procedures and software to allow use of earth edge corrections of line start timing errors.

The present procedures are being employed in processing data on the WINDCO (prototype McIDAS) system. We plan to employ somewhat different and more efficient procedures on the advanced McIDAS system. These will be discussed in subsequent reports.

Data Processing Activities

In response to needs of Center scientists, corrected digital tapes have been written for ATS-3 data from day 204, 1969 and day 203, 1969. The corrected images are 800 line swaths encompassing the target area of the experiment. We now have 16 such corrected images for day 204, ten based on left hand limb detection and six based on right hand limb detection. In addition, seven corrected images have been written for day 203, these based on left hand limb detection. Additional 203 data, as well as 202 data, will be processed during the next quarter.

Wind sets have now been obtained for both right hand and left hand based corrections. They show east-west residuals which are comparable to, or less than, those obtained from raw data with low line start timing errors. This preliminary verification that the correction procedures do result in significant data improvement will be made more quantitative by further tests planned for the next quarter.

Task B. Sun Glitter

Sun glitter studies have followed two paths: (1) an investigation of the utility and technique of obtaining sun glitter measurements from a geostationary satellite images, and (2) a study of the problems of geometry (pointing angle, etc.) from a near earth orbiting satellite.

Kornfield has almost completed a Ph.D. thesis on the first subject. He shows that one can indeed obtain excellent estimates of surface wind velocity (we already knew we could get good estimates of wind speed) from sun glitter measurements. There are two ways to get the wind direction information. One can use the mean square slope in the cross wind and the downwind directions separately. The sunglitter intensity is a measure of the slope probability--but it is also affected by the intervening atmosphere, scatter by the white caps, scatter by the ocean, and also glitter arising from the sky dome. He shows how to account for all of these and is able to get wind direction for typical conditions, but not the very light winds or very strong winds. He has also found--and this may be more important--that the displacement of the most probable slope; i.e., the brightest part of the sun glitter, from the position the sun's image would occupy if the sea were calm, is directly proportional to the square of surface wind. Indeed, it appears to be directly proportional to the wind stress on the sea surface. This kind of information is difficult to extract from geostationary satellite images because they cover such a wide area. This data would be easy to extract from near earth orbiting satellites. (Kornfield started this work with support from another sponsor, but his support until recently has been under this contract. Although no longer at the University, Kornfield is continuing to write his thesis, which must by University rules be submittedly by October 1973). There is no specific progress to report on item (2)--the geometry for the near earth satellite--except that the computer programming effort continues.

Task D. Cloud Growth Rate

A master's thesis outlining the requirements of a normalization procedure for ATS satellite picture has been completed. The draft Table of Contents and Introduction of the thesis is included below. In this work, using ATS-3 data the effects of changing sun-satellite-cloud geometry on the cloud brightness have been clearly demonstrated. One of the primary deficiencies of previous brightness normalization procedures has been that they apply the same correction factor for all parts of the cloud. The brightness contrast analyses within clouds are found to be a function of scattering angles. For a large cloud mass different parts of the same clouds appear to have different scattering properties and therefore may require different normalization corrections.

This study also looked into the observations of the scattering phase function and the effect of possible breaks in the clouds which were smaller than the camera resolution. The results of this investigation in the form of the completed thesis will be included in the annual report. Work is presently proceeding on developing a normalization procedure which will use a theoretical multiple scattering program to generate data on the brightness of clouds with various thickness for varying sun-cloud-satellite geometries.

PRELIMINARY

Table of Contents

I. Introduction

The purpose of this study is to look into the requirements of a normalization and standardization proceedure for satellite pictures. Meteorological satellites now provide pictures of the earth and its cloud cover on a routine basis. These satellites offer the coverage and resolution necessary to study and monitor mesoscale phenomèna such as convection. The satellites presently provide pretty pictures. However they offer the possibilities of providing considerablely more information on mesoscale phenomena than is presently obtainable. If one is to try to study mesocale phenomena from a satellite, one must first understand how light interacts with a cloud and what information about the clouds can be gotten from the light which the satellite receives.

The reflected light from clouds depends to a large extent on four variables:

a) Drop size distribution and phase state of water particles

b) Number density of scattering particles in the cloud

c) Cloud thickness

d) Angular condition of the measurement system (the zenith angles of the sun and sensor and their relative azimuth angle)

If one is to study some convective process, such as cloud growth, one wants only one variable to change at a time. If the cloud thickness changes, but the sun or the satellite also moves, one needs a method of removing the effect of the change in the measurement geometry. This is the normalization problem. Once the normalization problem has been satisfactorily solved, a considerable amount of information can be gotten from satellite photographs. The volume flux through a cloud can accurately be determined along with the associated rainfall and energy release. This combined with the extensive coverage offered by the satellite provides possibilities for exploring the large scale energetics associated with convective systems. But first the normalization problem must be solved.

There have been several approaches made at the normalization problem. The simplest approach has been to assume that clouds are perfect isotropic reflectors and obey Lambert's law. The intensity of the reflected light will vary as the cosine of the solar zenith angle for an unchanging cloud. There has been some evidence that this can be a valid assumption for very thick clouds. Martin and Suomi (72) have shown that the tops of cumulonimbus clouds display a Lambertian behavior. Sikdar and Suomi (72) also showed that thick clouds behave closely as Lambertian reflectors for small solar zenith angles ($+30^{o}$ to -30^{o}).

However there is considerable evidence in the literature that other clouds do not behave as Lambertian reflectors. Bartman(67) measured the scattering off stratocumulus clouds and found anisotropic reflectance. Ruff, et. at. (68) measured the angular distribution of solar radiation reflected from clouds as determined from TIROS IV radiometer measurements. They found clouds generally show an anisotropic reflectance pattern which varies with solar zenith angle. Brennan and Bandeen (70) measured the pattern of reflectance of solar radiation from stratocumulus clouds using an aircraft-born medium resolution radiometer. Their results show clouds to be anisotropic with the anisotropy increasing with increasing solar zenith angle.

Since there is ample evidence that clouds are not isotropic reflectors, Sikula and Vonder Haar (72) developed a normalization proceedure based on the empirical data of Brennan and Bandeen (70). This proceedure used empirical anisotropic factors which were tabulated for various solar zenith angles, satellite zenith angles, and the azmiuth angle between them. This proceedure is an improvement over the isotropic assumption, but it still has several drawbacks. One is that it needs continuous revision as more data becomes available. Another is that it makes no allowance for variations in the types and variations in the structure of clouds. For any given sun-cloud-satellite geometry, the same normalization factor is applied to all types of clouds, and to all parts of a single cloud. A cloud is a cloud, is a cloud; or is it?

The purpose of this study will be to look into the requirements of a better normalization system and to provide some suggestions as to possible methods of improving the normalization proceedures. In order to accomplish this, first the geometry of scattering and the definition of terms will be reviewed. Then the theoretical literature will be reviewed. The dependence of the single scattering phase function on the scatter's size distribution and shape will be briefly covered. For multiple scattering, the effects of the cloud's thickness, size distribution, and phase function shape will be covered. Then experimental observations of phase functions and of real clouds will be reviewed. In order to provide answers to questions raised by the theoretical and experimental literature review, I have made some observations. The question of does the contrast between the brightest parts and other parts of a cloud depend upon scattering geometry has not been answered by the experimental literature even though the theoretical literature suggests that the contrast does depend upon scattering geometry. This question is important for normalization because one must know if the same normalization factor can be applied to all parts of a single cloud, or if different factors must be applied to different parts. I will try to answer this question and also the question of what types of phase functions are found in clouds. Included in my analysis will be a discussion of possible effects, such as data biasing and the effect of possible inhomogeneties in the clouds which could cause misinterpretations of my results.

Task E. Comparative Studies in Satellite Stability

Debugging and testing of the computer coded stability model continues. Part VI, "Boundary Layer Response and Asymptotic Stability" is attached as an appendix to this progress report.

Task F. High Resolution Optics Study

In February 1973, NASA requested SSEC to reprogram part of its activities toward a study of potential meteorological uses of the Synchronous Earth Observation Satellite (SEOS). The basic purpose of the system would be to detect and predict hazards to life, property, or the quality of the environment, and to promote proper exploitation and conservation of resources. At that time, a seminar on mesoscale monitoring and prediction has been completed under the direction of Professor Suomi. Topics of study were chosen with an eye toward possible SEOS applications, but the students were interested mainly in what observables should be observed and in applying them to improving local weather communication to the general public.

In this seminar, the causes and behavior of tornadoes, hurricanes, heavy snow swaths, thunderstorms, hail, sleet, flash floods, clear air turbulence, fog, and lake and sea breezes were summarized. On the basis of these summaries the key observables in the motion, thermal, and moisture fields were tabulated. Individual charts were made and combined to form a composite chart identifying common observables. The information about the key observables to be monitored has been presented to SSEC project physicists and engineers in two days of briefings shortly before the end of the semester. The SSEC people will now consider how the measurements should be made and consider the impact of these requirements on possible sensing systems, especially SEOS.

As a result of post-briefing discussion, two primary conclusions are already evident:

1. SEOS in uniquely important, and virtually indispensible, for monitoring and study of the meteorological mesoscale. This is because it is the only observing platform combining the attributes of large area coverage, high resolution, high signal to noise ratio, and continual temporal coverage. Other observing platforms fail in one or more of the above to satisfy observing requirements.
2. SEOS will need a vertical temperature and moisture sounding capability. The combination of high resolution sounding and imaging is far more powerful than either alone since by knowing the temperature and moisture fields as well as the motion field one can get a handle on the dynamics of mesoscale activity as well as the kinematics. Knowing the forcing function permits inferences about future change, the key to prediction. Imaging alone tells us primarily what <u>has</u> happened, not what <u>may</u> happen.

<u>New Technology</u>

No items of new technology have been produced during this quarter.

<u>Program for next reporting interval</u>

<u>Task A.</u> <u>Investigations of Meteorlogical Data Processing Techniques</u>

The right and left edge correction procedure will be applied to tapes for day 202, 1969. The finding that this convection procedure does result in significant data improvement will be made more quantitative through tests beyond the simple comparisons of east-west residuals with residuals from raw data of low line start timing errors.

Additional tests will be preformed to establish relationships between the level in the atmosphere where brightness values indicate detection (limb radiance profiles) and sun satellite geometry, clouds at the limb at the place of scan, and latitude.

Task B. Sun Glitter

This task has been curtailed to provide funds for the new Task F. Kornfield's completed Ph.D. thesis should be available by October 1973; in any case the work will be made available as a final report.

Task D. Cloud Growth Rate

Work will begin on a new brightness normalization procedure which will use a theoretical multiple scattering program to generate data on cloud brightness as a function of thickness and varying sun-cloud-satellite geometries.

Task E. Comparative Studies in Satellite Stability

Debugging and testing of the computer coded model will continue.

Task F. High Resolution Optics Study

In addition to the summary charts prepared by the mesoscale monitoring seminar mentioned above (charts which stress the meteorological factors), a systematic classification matrix is being assembled in order to optimize the specific technical requirements which must be met to successfully monitor the meteorological observables. Measurement Requirement sheets such as the sample attached will be filled out and the information transferred to punched cards and computer correlated and classified to develop a consistent set of technical

PLEASE USE PENCIL

MEASUREMENT REQUIREMENT

☐ MAJOR PROBLEM OR NEED

- SCIENTIFIC VALUE → ☐
- ECONOMIC VALUE → ☐
- SOCIAL VALUE → ☐
- OTHER VALUE → ☐

20 ☐ INVESTIGATOR

DON'T KNOW · NO VALUE · LOW · HIGH · HIGHLY SIGNIFICANT (ONLY SOURCE)

0 1 2 3 4 5 6 7 8 9

VALUE SCALE

☐ KNOWLEDGE REQUIREMENT

└VALUE → ☐ → (IN SATISFYING MAJOR PROBLEM)

☐ OBSERVATION REQUIREMENT

└VALUE → ☐ → (IN SATISFYING KNOWLEDGE REQUIREMENT)

HYPOTHESIS · PROVEN THEORY · MEASURED IN LAB · MEASURED IN FIELD · MEASURED REMOTELY

1 2 3 4 5

DEVELOPMENT STATUS

PLATFORM USED ☐

1- SEOS
2- Geosynchronous Satellite
3- Low Orbit Satellite
4- Airplane
5- Balloon
6- Ship
7- Buoy
8- Land Based
9- Other: ________

SPECTRAL BAND ☐

1- UV < 0.35 μm
2- Visible .35 - .75 μm
3- Near IR .7 - 3.5 μm
4- Thermal IR 3.5 - 20 μm
5- Far IR 20 μm - 100 μm
6- Microwave .1 mm - 50 cm
7- Radar, TV, .5 - 100 m
8- Long Wave > .1 km
9- P, T, other: ________

MEASUREMENT PARAMETERS: (PLEASE USE STANDARD UNITS)

24 S/N 29

1- BAND CENTER: ________ 30 37 39

2- BAND WIDTH: ________ 40 47 49

3- TEMPORAL FREQ.: ________ 50 57 59

4- SPATIAL RESOL.: ________ 60 67 69

5- S/N & RMS ERROR: S/N % ‖ RMS 70 RMS 77 79 80

6- ☐ AREA COVERAGE KM → 100 (LOCAL) 500 (MESOSCALE) 1000 (SYNOPTIC) 5000 10000 (GLOBAL)

1 2 3 4

SPECIAL REQUIREMENTS: ________

7- ☐ TIME COVERAGE <1d. 1d - 1 mo. 1-12 mo. 1-10 y. >10 y.

1 2 3 4 5

8- ☐ MEAS. DEVELOPMENT STATUS

1 4 6 9 10 11 15 17 18 19 20 21 25

MEASUREMENT IS:

☐ PRIMARY
☐ SUPPORTING

21

PLATFORM IS:

☐ PART OF SYSTEM
☐ COMPLETELY SEPARATE UNIT

22

MEASUREMENT IS:

☐ UNIQUE TO SEOS
☐ NOT UNIQUE

23

requirements which is complete for the phenomena we have investigated.

If time permits, the study will be broadened to other mesoscale phenomena and potential uses of SEOS in order to make a more complete survey. For the present, we are concentrating on analyzing and documenting the unique and significant contributions of the seminar. A full description of the study results will be completed before the next quarterly report.

Conclusions and Recommendations

None at this time.

APPENDIX

Part VI

"Boundary Layer Response and Asymptotic Stability"

1. Introduction:

In the following, the term "Boundary Layer Response" refers to the motion of a flexible satellite near the time t = 0, and is not to be confused with the response of the atmosphere.

In the earlier phases of this work, it was seen that the response of the system near the time t = 0 was found to be given essentially by the outer boundary layer equations (3.70) and (3.71). These are rewritten here as follows:

$$A(\underline{\omega}_B,0)\ddot{\underline{y}}(\underline{\omega}_B,\tau) + \sqrt{\varepsilon}\, B(\underline{\omega}_B,0)\dot{\underline{y}}(\underline{\omega}_B,\tau) + \underline{y}(\underline{\omega}_B,\tau) = \underline{0} \qquad (3.70;\ 6.1)$$

$$[A\nabla\underline{y}](\underline{\omega}_B,0)\ddot{\underline{\omega}}_1(\tau) + 2[A\nabla\dot{\underline{y}}](\underline{\omega}_B,0)\dot{\underline{\omega}}_1(\tau)$$

$$+ \underline{\nabla}[A\ddot{\underline{y}}+\underline{y}](\underline{\omega}_B,0)\underline{\omega}_1 = \underline{d}(\underline{\omega}_B,0) \qquad (3.71;\ 6.2)$$

It was also seen that the long time response of the system is basically governed by Eqs. (5.58) and (5.113). These equations are

$$B_1\ddot{\underline{x}} + B_2\dot{\underline{x}} + B_3\underline{x} = 0 \qquad (5.58;\ 6.3)$$

and

$$A_1\ddot{\underline{q}}_o + A_{20}\dot{\underline{q}}_o + A_{30}\underline{q}_o = \underline{A}_{40} \qquad (5.113;\ 6.4)$$

Equations (6.1) and (6.2) will now be considered in detail for the boundary layer response. Then Eqs. (6.1) through (6.4) will be examined for stability of the satellite motion. The notation in these equations were defined before.

2. The Boundary Layer Equations

The functional matrices A and B of Eqs. (6.1) and (6.2) are defined in Eqs. (3.6) and (3.7). In Eqs. (6.1) and (6.2) these matrices are to be evaluated at $t = 0$. This makes these two linear second order equations with constant coefficients. Hence $\underline{y}(\tau)$ is given by

$$\underline{y}(\underline{\tau}) = \phi_3\underline{y}(0) + \phi_4\dot{\underline{y}}(0) \tag{6.5}$$

where ϕ_3 and ϕ_4 are fundamental matrices of Eq. (6.1). Thus $\underline{y}(\tau)$ undergoes damped oscillations. For the two bodies A and B, Eq. (6.5) becomes

$$\underline{M}_{A3} = \phi_{A3}\,\underline{M}_{A3}(0) + \phi_{A4}\,\dot{\underline{M}}_{A3}(0) \tag{6.6}$$

$$\underline{M}_{B3} = \phi_{B3}\,\underline{M}_{B3}(0) + \phi_{B4}\,\dot{\underline{M}}_{B3}(0) \tag{6.7}$$

Substituting the forms of $\underline{M}_{A3}$ and $\underline{M}_{B3}$ from Eqs. (4.21) and (4.23) into Eqs. (6.6) and (6.7), respectively, we obtain

$$\begin{aligned}
& S_{A1}\dot{\underline{\omega}}_B + S_{A1}\ddot{\underline{\theta}} + \Big\{ 2S_{A1}\tilde{\omega}_B + S_{A2} + S_{A3} + S_{A5}\tilde{\omega}_B\delta_{A2} \\
& - S_{A5}\widetilde{[\delta_{A2}\underline{\omega}_B]} + \overset{*}{\delta}_A\omega_B I_A - \overset{*}{\delta}_A\widetilde{[I_A\omega_B]}\Big\}\dot{\underline{\theta}} + \Big\{ S_{A1}(\dot{\tilde{\omega}}_B \\
& + \tilde{\omega}_B\tilde{\omega}_B) + S_{A2}\tilde{\omega}_B + S_{A4} + S_{A5}\tilde{\omega}_B\delta_{A2}\tilde{\omega}_B - S_{A5}\widetilde{[\delta_{A2}\underline{\omega}_B]}\tilde{\omega}_B \\
& + \overset{*}{\delta}_A\tilde{\omega}_B I_A\tilde{\omega}_B - \overset{*}{\delta}_A\widetilde{[I_A\underline{\omega}_B]}\tilde{\omega}_B\Big\}\underline{\theta} + S_{A2}\underline{\omega}_B \\
& + [S_{A5}\dot{\tilde{\theta}}\delta_{A2}\dot{\underline{\theta}} + \overset{*}{\delta}_A\dot{\tilde{\theta}}I_A\dot{\underline{\theta}}] + \Big\{ S_{A5}\dot{\tilde{\theta}}\delta_{A2}\tilde{\omega}_B\underline{\theta} \\
& + S_{A5}\widetilde{[\tilde{\omega}_B\underline{\theta}]}\delta_{A2}\dot{\underline{\theta}} + \overset{*}{\delta}_A\dot{\tilde{\theta}}I_A\tilde{\omega}_B\underline{\theta} + \overset{*}{\delta}_A\widetilde{[\tilde{\omega}_B\underline{\theta}]}I_A\dot{\underline{\theta}}\Big\} \\
& + \Big\{ S_{A5}\widetilde{[\tilde{\omega}_B\underline{\theta}]}\delta_{A2}\tilde{\omega}_B\underline{\theta} + \overset{*}{\delta}_A\widetilde{[\tilde{\omega}_B\underline{\theta}]}I_A\tilde{\omega}_B\underline{\theta}\Big\} \\
& + \Big\{ S_{A5}\tilde{\omega}_B\delta_{A2}\underline{\omega}_B + \overset{*}{\delta}_A\tilde{\omega}_B I_A\underline{\omega}_B\Big\} - \overset{*}{\delta}_A\underline{u}_A \\
& \quad = \phi_A\,\underline{M}_{A3}(0) + \phi_{A4}\dot{\underline{M}}_{A3}(0)
\end{aligned} \tag{6.8}$$

20

and

$$S_{B1}\dot{\underline{\omega}}_B + S_{B2}\dot{\underline{\theta}} + S_{B3}\underline{\omega}_B + S_{B4}\underline{\theta} + S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B$$

$$+\delta_B\overset{*}{\tilde{\omega}}_B I_B\underline{\omega}_B - \overset{*}{\delta}_B\underline{u}_B = \phi_{B3}\underline{M}_{B3}(0) + \phi_{B4}\dot{\underline{M}}_{B3}(0) \tag{6.9}$$

From Eq. (6.9), $\underline{\theta}$ is solved in terms of $\underline{\omega}_B$, and we obtain

$$\underline{\theta} = \exp[-S_{B2}^{-1}S_{B4}t] \left\{ \overset{*}{\underline{\theta}} + \int_0^t \exp[S_{B2}^{-1}S_{B4}\tau] \; [\underline{g}(\tau) + \phi_{B3}\underline{M}_{B3}(0) + \phi_{B4}\dot{\underline{M}}_{B3}(0)] \right\} \tag{6.10}$$

Eliminating $\underline{\theta}$, $\dot{\underline{\theta}}$ and $\ddot{\underline{\theta}}$ from Eq. (6.8) by using Eqs. (6.9) and (6.10), the boundary layer equation for $\underline{\omega}_B$, corresponding to Eq. (4.6a) is obtained as

$$B_1\ddot{\underline{\omega}}_B + B_2\dot{\underline{\omega}}_B + B_3\underline{\omega}_B + B_4\left\{ \int_0^t B_5(\tau-t) \; [\overset{*}{\delta}_B\underline{u}_B - S_{B1}\dot{\underline{\omega}}_B - S_{B3}\underline{\omega}_B](\tau)\,d\tau \right\} + B_6\dot{\underline{u}}_B + B_7\underline{u}_B - \overset{*}{\delta}_A\underline{u}_A$$

$$+\varepsilon'\underline{f}(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{\theta}, \underline{u}_A, \underline{u}_B) = B_8(t) \tag{6.11}$$

where

$$B_8(t) = -B_4F_1(t)\overset{*}{\underline{\theta}} - \left[S_{A1}S_{B2}^{-1}\dot{\phi}_{B3}(t) + (S_{A2}+S_{A3})S_{B2}^{-1}\phi_{B3}(t) \right] \underline{M}_{B3}(0)$$

$$- \left[S_{A1}S_{B2}^{-1}\dot{\phi}_{B4}(t) + (S_{A2}+S_{A3})S_{B2}^{-1}\phi_{B4}(t) \right] \dot{\underline{M}}_{B3}(0)$$

$$-S_{A4}\int_0^t B_5(\tau-t) \left[\phi_{B3}(\tau)\underline{M}_{B3}(0) + \phi_{B4}(\tau)\dot{\underline{M}}_{B3}(0) \right] d\tau$$

$$+\phi_A(t)\underline{M}_{A3}(0) + \phi_{A4}(t)\dot{\underline{M}}_{A3}(0). \tag{6.12}$$

As the body B is nominally static, the terms involving the products of $\underline{\omega}_B$ or $\dot{\underline{\omega}}_B$ with $\underline{M}_{B3}(t)$ have been neglected in deriving Eq. (6.11).

3. Boundary Layer Solution for $\underline{\omega}_B$

The maximal probabilistic boundary layer solution for $\underline{\omega}_B$ is obtained in a way similar to that used for the asymptotic case. Corresponding to Eqs. (5.44) and (5.45), the governing equations in this case become

$$B_1^T\ddot{\underline{\lambda}}_1 - B_2^T\dot{\underline{\lambda}}_1 + B_3^T\underline{\lambda}_1 = \varepsilon'\left[g_1\dot{\underline{\lambda}}_1 + g_2\underline{\lambda}_1\right] \tag{6.13}$$

and

$$B_1\ddot{\underline{\omega}}_B + B_2\dot{\underline{\omega}}_B + B_3\underline{\omega}_B - B_4\left\{\int_0^t B_5(\tau-t)\left(S_{B1}\dot{\underline{\omega}}_B + S_{B3}\underline{\omega}_B\right)(\tau)\,d\tau\right\}$$

$$+\bar{\underline{\omega}}(t) - B_8(t) = \varepsilon'\left[\int_0^t g_3(t_1\tau)\underline{\lambda}_1(\tau)\,d\tau + g_4\dot{\underline{\lambda}}_1 + g_5\underline{\lambda}_1 - \underline{f}\right] \tag{6.14}$$

Comparing Eqs. (6.13) and (6.14) with Eqs. (5.44) and (5.45) we see that the term $B_4F_1\underline{\theta}^*$ of Eq. (5.45) has been replaced by $-B_8(t)$ in Eq. (6.14) without any other change. Hence all the expressions for $\underline{\omega}_B$, derived for the asymptotic case becomes applicable to the boundary layer case if $-B_8(t)$ is substituted for $B_4F_1\underline{\theta}^*$.

4. Boundary Layer Solution for $\underline{\omega}_1$

The governing equation for the boundary layer solution for $\underline{\omega}_1$ is given by Eq. (6.2). As the coefficients of $\underline{\omega}_1$, $\dot{\underline{\omega}}_1$ and $\ddot{\underline{\omega}}_1$ are evaluated at $t = 0$, the equation is one with constant coefficients. The initial conditions are lumped on Eq. (6.1) and so $\underline{\omega}_1$ has zero initial conditions. Hence the solution $\underline{\omega}_1$ for the bodies A and B are given respectively by the Duhamel's integrals as

$$\underline{\omega}_{A1} = \int_0^t \phi_{A5}(t-\tau)\,\underline{d}(\underline{\omega}_B, 0)\,d\tau \tag{6.15}$$

and

$$\underline{\omega}_{B1} = \int_0^t \phi_{B5}(t-\tau)\underline{d}(\underline{\omega}_B,0)\,d\tau. \qquad (6.16)$$

In the above two equations, ϕ_{A5} and ϕ_{B5} are the appropriate fundamental matrices of the homogeneous equations of Eq. (6.2) for the bodies A and B, respectively.

5. Stability Criteria

The basic requirement for a stable satellite motion is that the angular velocities $\omega_{B,1}$ and $\omega_{B,2}$ should approach zero values as $t \to \infty$. The components $\underline{x}_i$, $(i=0,1,2,\ldots)$, of $\underline{\omega}_B$ are given by Eqs. (5.47) and (5.51) through (5.56). It is seen that the forms of the solutions for $\underline{x}_i$ are governed by the roots of the homogeneous equation (5.58). For convenience, Eq. (5.58) is repeated here as

$$B_1\ddot{\underline{x}}_i' + B_2\dot{\underline{x}}_i' + B_3\underline{x}_i' = 0 \qquad (5.58;\ 6.17)$$

For asymptotic stability, the roots of this equation must have negative real parts. The characteristic equation of Eq. (6.17) is

$$\begin{aligned}
&p^6 + \mathrm{tr}\cdot[B_1^{-1}B_2]p^5 + \left\{\mathrm{tr}\cdot[B_1^{-1}B_3] + \mathrm{Det}[B_1^{-1}B_2]\mathrm{tr}\cdot[B_2^{-1}B_1]\ p^4\right\} \\
&+\mathrm{Det}[B_1^{-1}]\left\{\mathrm{Det}[B_2] + e_{rst}e_{ijk}B_{3,ri}B_{1,sj}B_{2,tk}\right\}p^3 + \left\{\mathrm{Det}\cdot[B_1^{-1}B_3]\mathrm{tr}\cdot[B_3B_1]\right. \\
&\left.+\mathrm{Det}\cdot[B_1^{-1}B_2]\mathrm{tr}\cdot[B_2^{-1}B_3]\right\}p^2 + \left\{\mathrm{Det}\cdot[B_1^{-1}B_3]\mathrm{tr}\cdot[B_3^{-1}B_2]\right\}p \\
&+\mathrm{Det}[B_1^{-1}B_3] = 0 \qquad (6.18)
\end{aligned}$$

In Eq. (6.18), e_{ijk} and e_{rst} are the permutation symbols.

The Routh series for Eq. (6.18) is now calculated and hence we obtain the following inequalities [1] to be satisfied for stability of the linear

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core of Eq. (4.46).

$$\mathrm{tr}\cdot[B_1^{-1}B_2] > 0 \tag{6.19}$$

$$\mathrm{tr}\cdot[B_1^{-1}B_2]\cdot\left\{\mathrm{tr}\cdot[B_1^{-1}B_3] + \mathrm{Det}\cdot[B_1^{-1}B_2]\mathrm{tr}\cdot[B_2^{-1}B_1]\right\}$$
$$-\mathrm{Det}[B_1^{-1}]\left\{\mathrm{Det}[B_2] + e_{rst}e_{ijk}B_{3,ri}B_{1,sj}B_{2,tk}\right\} > 0 \tag{6.20}$$

$$\mathrm{Det}[B_1^{-1}]\cdot\left\{\mathrm{tr}\cdot[B_1^{-1}B_2]\right\}\left\{\mathrm{tr}\cdot[B_1^{-1}B_3] + \mathrm{Det}[B_1^{-1}B_2]\mathrm{tr}\cdot[B_2^{-1}B_4]\right\}\left\{\mathrm{Det}[B_2]\right.$$
$$\left.+e_{rst}e_{ijk}B_{3,ri}B_{1,sj}B_{3,tk}\right\} - \left[\mathrm{Det}(B_1^{-1}]\left\{\mathrm{Det}[B_2] +\right.\right.$$
$$\left.\left.e_{rst}e_{ijk}B_{3,ri}B_{1,sj}B_{3,tk}\right\}\right]^2 - \left\{\mathrm{tr}\cdot[B_1^{-1}B_2]\right\}^2\left\{\mathrm{Det}\cdot[B_1^{-1}B_3]\mathrm{tr}\cdot[B_3^{-1}B_1]\right.$$
$$\left.+\mathrm{Det}\cdot[B_1^{-1}B_2]\mathrm{tr}\cdot[B_2^{-1}B_3]\right\} + \left\{\mathrm{tr}\cdot[B_1^{-1}B_2]\right\}\left\{\mathrm{Det}[B_1^{-1}B_3]\mathrm{tr}\cdot[B_3^{-1}B_2]\right\} > 0 \tag{6.21}$$

Rewriting Eq. (6.18) as

$$p^6 + a_1p^5 + a_2p^4 + a_3p^3 + a_4p^2 + a_5p + a_6 = 0 \tag{6.22}$$

the fourth in equality is obtained as

$$k = (a_1a_2a_3a_4 + a_2a_3a_5 - a_3^2a_4 - a_1^2a_4^3 - a_5^2 + 2a_1a_4a_5$$
$$- a_1a_2^2a_5 - a_1^2a_2a_6 - a_1a_3a_6) > 0 \tag{6.23}$$

The fifth inequality is given by

$$[k(a_1a_2a_5 - a_3a_5 - a_1^2a_6) + 2a_1a_3a_6b_1b_3 - a_3^2b_1^2a_6 - a_1b_3^3a_6] > 0 \tag{6.24}$$

where

$$b_1 = a_1a_2 - a_3 \tag{6.25}$$

and

$$b_3 = a_1a_4 - a_5. \tag{6.26}$$

The last inequality to be satisfied is

$$a_6 = \mathrm{Det}\cdot[B_1^{-1}B_3] > 0 \tag{6.27}$$

The above inequalities will all be obviously satisfied in a well-designed satellite. These also yield a comparison technique for two satellites. The comparison is to be made on marginal stability. The design for which a_1 and a_6 are larger has a larger marginal stability and hence is the more desirable one.

We now consider Eq. (4.25) to examine the conditions for stability of the relative rotation, $\underline{\theta}$. Equation (4.25) is repeated here for ease of reference.

$$\underline{\theta} = \exp\left[-S_{B2}^{-1}S_{B4}t\right]\underline{\theta}^{*} + \int_0^t \exp\left[S_{B2}^{-1}S_{B4}(\tau-t)\right]\underline{g}(\tau)\,d\tau \tag{4.25; 6.28}$$

where $g(\tau)$ is defined by Eq. (4.26). Evidently, the primary requirement for stability of $\underline{\theta}$ is that the eigenvalues of the matrix $[-S_{B2}^{-1}S_{B4}]$ must have non-positive real parts. There is another interesting requirement on these eigenvalues. This arises due to the appearance of $g(\tau)$ in the integrand in Eq. (6.28). This requirement is obtained in the following discussion.

Let $-\mu_1$, $-\mu_2$ and $-\mu_3$ be the real parts of the eigenvalues of $[-S_{B2}^{-1}S_{B4}]$, such that

$$|\mu_1| > |\mu_2| > |\mu_3| \tag{6.29}$$

Let $-\sigma_1, -\sigma_2, -\sigma_3$ be the real parts of the eigenvalues of Eq. (6.17), such that

$$|\sigma_1| > |\sigma_2| > |\sigma_3| \,. \tag{6.30}$$

Then from Eq. (6.28), we have

$$|\underline{\theta}| \leq \hat{\theta}e^{-\mu_3 t} + \int_0^t \alpha_1 \cdot e^{\mu_1\tau} \cdot e^{-\mu_3 t} \cdot e^{-\sigma_3\tau}\,d\tau$$

or

$$|\underline{\theta}| \leq \hat{\theta}e^{-\mu_3 t} + \frac{\alpha_1}{\mu_1-\sigma_3} e^{-(\sigma_3+\mu_3-\mu_1)t} \tag{6.31}$$

Hence for $|\underline{\theta}|$ to be asymptotically stable, we must have

$$\sigma_i \neq \mu_i, \; i=1,2,3. \tag{6.32a}$$

$$\sigma_3 \geq (\mu_1-\mu_3). \tag{6.32b}$$

The variables $\hat{\theta}$ and α_1 introduced in Eq. (6.31) are suitable finite constants.

We now consider Eq. (5.60) for the stability of the component $\underline{x}_o'(t)$ of $\underline{\omega}_B(t)$. For convenience, Eq. (5.60) is rewritten below.

$$\underline{x}_o'(t) = \phi_1(t)\underline{b}_{10} + \phi_2(t)\underline{b}_{20} - \int_0^t \phi_2(t-\tau)B_4 \left\{ F_1(\tau)\underline{\theta}^* - \int_0^t B_5(S-\tau)[S_{B1}\dot{\underline{x}}_o'(0) + S_{B3}\underline{x}_o'(0)]ds \right\} d\tau - \int_0^t \phi_2(t-\tau)\underline{w}_o(\tau)d\tau. \tag{5.60; 6.33}$$

Since the first two terms and the last integral on the right hand side of Eq. (6.33) tend to zero as $t \to \infty$, the truncated form of Eq. (6.33) given below is considered.

$$\underline{x}_o'(t) \doteq -\int_0^t \phi_2(t-\tau)B_4 \left\{ F_1(\tau)\underline{\theta}^* - \int_0^t B_5(S-\tau)\left[S_{B1}\dot{\underline{x}}_o'(0)+S_{B3}\underline{x}_o'(0)\right] ds \right\} d\tau.$$

Going through the derivation of Eq. (6.33) it can be seen that the above equation is generated from

$$\underline{x}_o'(t) \doteq -\int_0^t \phi_2(t-\tau)B_4\underline{\theta}(\tau)d\tau \tag{6.34}$$

Hence using Eq. (6.31), Eq. (6.34) is converted to

$$|\underline{x}_o'(t)| \le \alpha_2 \cdot \int_0^t e^{-\sigma_3 t} \cdot e^{\sigma_1 \tau} \left[e^{-\mu_3 \tau} + \alpha_3 e^{(\mu_1 - \mu_3 - \sigma_3)\tau} \right] d\tau$$

or

$$|\underline{x}_o'(t)| \le \frac{\alpha_2}{\sigma_1 - \mu_3} \cdot e^{(\sigma_1 - \mu_3 - \sigma_3)t} + \frac{\alpha_2 \alpha_3}{(\sigma_1 + \mu_1 - \sigma_3 - \mu_3)} e^{(\sigma_1 + \mu_1 - \mu_3 - 2\sigma_3)t} \tag{6.35}$$

From Eq. (6.35), the necessary stability conditions are

$$\sigma_1 \le \sigma_3 + \mu_3 \tag{6.36}$$

$$\sigma_1 \le 2\sigma_3 + \mu_3 - \mu_1 \tag{6.37}$$

Inequalities (6.32b), (6.36) and (6.37) can be combined to obtain the following system of inequalities.

$$\sigma_1 \le (\sigma_3 + \mu_3); \quad \sigma_3 > \mu, \tag{6.38}$$

or

$$\sigma_1 \le (2\sigma_3 + \mu_3 - \mu_1); \quad \mu_1 > \sigma_3 > (\mu_1 - \mu_3) \tag{6.39}$$

If the above inequalities are satisfied, both $\underline{\theta}$ and $\underline{x}_o'$ tend to zero asymptotically. It is to be noted that $\underline{x}_o'(t)$, given by Eq. (6.33), is also the solution of the linear core of the deterministic equation (4.46a). Using this fact, we now prove that the non-linear equation (4.46a) is stable at the origin.

The non-linear term in Eq. (4.46a) is $\varepsilon' \underline{f}(\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B)$. Comparing Eqs. (4.46a) and (4.35), it can be seen that $\varepsilon' \underline{f}$ consists of quadratic and higher order terms of its arguments. Since

$$\underline{u}_A, \underline{u}_B \to 0$$

if

$$\underline{\omega}_B, \dot{\underline{\omega}}_B \to 0 \ ,$$

we have

$$\text{Lim} \frac{|\varepsilon f'(v)|}{|\underline{v}|} = 0 \qquad (6.40)$$
$$\underline{\omega}_B \to 0$$
$$\dot{\underline{\omega}}_B \to 0$$
$$\underline{v} = \left[\dot{\underline{\omega}}_B, \underline{\omega}_B, \underline{u}_A, \underline{u}_B\right]^T .$$

Then from the Theorem 1, of Struble [2], and the non-linearity condition (6.40), it follows that Eq. (4.46a) is stable at $\underline{v}(t) = 0$.

We now analyze the behaviour of the stochastic differential equations (5.44) and (5.45). These non-linear equations are

$$B_1^T\ddot{\underline{\lambda}}_1 - B_2^T\dot{\lambda} + B_3^T\underline{\lambda}_1 = \varepsilon'\left[g_1\dot{\underline{\lambda}}_1 + g_2\underline{\lambda}_1\right] \qquad (5.44;\ 6.41)$$

$$B_1\ddot{\underline{\omega}}_B + B_2\dot{\underline{\omega}}_B + B_3\underline{\omega}_B - B_4\left\{\int_0^t B_5(\tau - t)(S_{B1}\dot{\underline{\omega}}_B + S_{B3}\underline{\omega}_B)(\tau)\,d\tau\right\}$$
$$+\underline{\bar{w}}(t) + B_4F_1\underline{\theta}^* = \varepsilon'\left[\left[\int_0^t g_3(t,\tau)\lambda_1(\theta)\,d\tau + g_4\dot{\underline{\lambda}}_1 + g_5\underline{\lambda}_1 - \underline{f}\right] \qquad (5.45;\ 6.42)$$

It can be proved that the eigenvalues of Eq. (6.17) and those of the left hand side of Eq. (6.41) are equal in magnitude but opposite in sign. Thus $\underline{\lambda}_1$ is unbounded. This result is expected as $\underline{\lambda}_1$, which is a measure of the indeterminacy of the value of $\underline{\omega}_B$, should grow larger with time. An interesting point to be noted is that the larger the marginal stability of $\underline{\omega}_B$, the larger is the marginal instability of $\underline{\lambda}_1$. Equation (6.42) is now considered to resolve this apparent paradox.

As it stands, Eq. (6.42) gives unbounded solutions for $\underline{\omega}_B$. This can be seen from the following. Let only the term $g_5\underline{\lambda}_1$ be considered on the right hand side of Eq. (6.42). The solution corresponding to this term is given by

$$\underline{x}(t) = \varepsilon' \int_0^t \phi_2(t-\tau) g_5(t) \underline{\lambda}_1(\tau) d\tau \tag{6.43}$$

Since

$$g_5(t) = \left[B_6 \frac{d}{dt} + B_7\right] \left[U_B \left(\frac{\partial \underline{f}}{\partial \underline{u}_B} - \frac{d}{dt} \frac{\partial \underline{f}}{\partial \dot{\underline{u}}_B}\right)^T\right] \overset{*}{\delta}_A U_A \frac{\partial \underline{f}}{\partial \underline{u}_A} ,$$

$g_5(t)$ behaves like $e^{-\sigma_3 t}$.

Hence

$$|\underline{x}(t)| \leq k\varepsilon' \int_0^t e^{-\sigma_3 t + \sigma_1 \tau} \cdot e^{(\sigma_1 - \sigma_3)\tau} d\tau$$

or

$$|\underline{x}(t)| \leq k_1 e^{2(\sigma_1 - \sigma_3)t} \tag{6.44}$$

where k and k_1 are suitable finite constants. Thus from Eqs. (6.30) and (6.44),

$$|\underline{x}(t)| \to \infty \quad \text{as} \quad t \to \infty.$$

To have bounded solutions for $\underline{\omega}_B$, we must have either

$$\sigma_1 = \sigma_2 = \sigma_3 = 0 \tag{6.45}$$

or

$$U_A = U_B = 0 \tag{6.46}$$

or

$$\text{Det}\cdot\left[\frac{\partial \underline{f}}{\partial \underline{u}_B} - \frac{d}{dt} \frac{\partial \underline{f}}{\partial \dot{\underline{u}}_B}\right] = \text{Det}\cdot\left[\frac{\partial \underline{f}}{\partial \underline{u}_A}\right] = 0 \tag{6.47}$$

The condition (6.45) ensures purely oscillatory motion of the satellite and is too restrictive. The condition (6.46) implies deterministic forcing and control torques. This also is quite difficult to achieve. The implications of the third condition (6.47) will be analyzed now.

Expanding the non-linear terms of Eq. (5.45), it is seen that $\underline{f}$ does not contain $\underline{\dot{u}}_B$ and $\underline{u}_A$. Hence

$$\mathrm{Det}\cdot\left[\frac{\partial \underline{f}}{\partial \underline{u}_A}\right] \equiv 0 \tag{6.48}$$

$$\frac{\partial \underline{f}}{\partial \underline{\dot{u}}_B} \equiv 0 \tag{6.49}$$

We also have

$$
\begin{aligned}
\frac{\partial \underline{f}}{\partial \underline{u}_B} = - \Bigg[& 2S_{A1}\tilde{\omega}_B + S_{A5}\tilde{\omega}_B\delta_{A2} - S_{A5}\widetilde{(\delta_{A2}\underline{\omega}_B)} + \overset{*}{\delta}_A\tilde{\omega}_B I_A \\
& -\overset{*}{\delta}_A\widetilde{(I_A\underline{\omega}_B)}\Bigg] S_{B2}^{-1}\overset{*}{\delta}_B + S_{A5}\Bigg[\widetilde{(S_{B2}^{-1}S_{B1}\underline{\dot{\omega}}_B)} + \widetilde{(S_{B2}^{-1}S_{B3}\underline{\omega}_B)} \\
& +\widetilde{(S_{B2}^{-1}S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B)} + \widetilde{(S_{B2}^{-1}\overset{*}{\delta}_B\tilde{\omega}_B I_B\underline{\omega}_B)} - \widetilde{(S_{B2}^{-1}\overset{*}{\delta}_B\underline{u}_B)} \\
& +\widetilde{(S_{B2}^{-1}S_{B4}\underline{\theta})}\Bigg]\delta_{A2}S_{B2}^{-1}\overset{*}{\delta}_B + S_{A5}\Bigg[\widetilde{(\delta_{A2}S_{B2}^{-1}S_{B1}\underline{\dot{\omega}}_B)} + \widetilde{(\delta_{A2}S_{B2}^{-1}S_{B3}\underline{\omega}_B)} \\
& +\widetilde{(\delta_{A2}S_{B2}^{-1}S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B)} + \widetilde{(\delta_{A2}S_{B2}^{-1}\overset{*}{\delta}_B\tilde{\omega}_B I_B\underline{\omega}_B)} - \widetilde{(\sigma_{A2}S_{B2}^{-1}\overset{*}{\delta}_B\underline{u}_B)} \\
& +\widetilde{(\delta_{A2}S_{B2}^{-1}S_{B4}\underline{\theta})}\Bigg] S_{B2}^{-1}\overset{*}{\delta}_B + \overset{*}{\delta}_A\Bigg[\widetilde{(S_{B2}^{-1}S_{B1}\underline{\dot{\omega}}_B)} + \widetilde{(S_{B2}^{-1}S_{B3}\underline{\omega}_B)} \\
& +\widetilde{(S_{B2}^{-1}S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B)} + \widetilde{(S_{B2}^{-1}\overset{*}{\delta}_B\tilde{\omega}_B I_B\underline{\omega}_B)} - \widetilde{(S_{B2}^{-1}\overset{*}{\delta}_B\underline{u}_B)} \\
& +\widetilde{(S_{B2}^{-1}S_{B4}\underline{\theta})}\Bigg] I_A S_{B2}^{-1}\overset{*}{\delta}_B + \overset{*}{\delta}_A\Bigg[\widetilde{(I_A S_{B2}^{-1}S_{B1}\underline{\dot{\omega}}_B)} + \widetilde{(I_A S_{B2}^{-1}S_{B3}\underline{\omega}_B)}
\end{aligned}
$$

$$
\begin{aligned}
&+(\widetilde{I_A S_{B2}^{-1} S_{B5} \tilde{\omega}_B \delta_{B2} \underline{\omega}_B}) + (\widetilde{I_A S_{B2}^{-1} \overset{*}{\delta}_B \tilde{\omega}_B I_B \underline{\omega}_B}) - (\widetilde{I_A S_{B2}^{-1} \overset{*}{\delta}_B \underline{u}_B}) \\
&+(\widetilde{I_A S_{B2}^{-1} S_{B4} \underline{\theta}})\Big] S_{B2}^{-1} \overset{*}{\delta}_B - S_{A5}(\widetilde{\delta_{A2} \tilde{\omega}_B \underline{\theta}}) S_{B2}^{-1} \overset{*}{\delta}_B - S_{A5}(\widetilde{\tilde{\omega}_B \underline{\theta}}) \delta_{A2} S_{B2}^{-1} \overset{*}{\delta}_B \\
&-\overset{*}{\delta} A(\widetilde{I_A \tilde{\omega}_B \underline{\theta}}) S_{B2}^{-1} \overset{*}{\delta}_B - \overset{*}{\delta}_A (\widetilde{\tilde{\omega}_B \underline{\theta}}) I_A S_{B2}^{-1} \overset{*}{\delta}_B
\end{aligned}
\tag{6.50}
$$

From the above equation, it is seen that Eq. (6.47) can be satisfied if

$$
\mathrm{Det}\left[S_{B2}^{-1} \overset{*}{\delta}_B \right] = 0 \tag{6.51}
$$

Since Eq. (6.51) can never be satisfied, it seem that the random torques tend to increase the variances of $\underline{\omega}_B$ indefinitely though the mean values of $\underline{\omega}_B$ decrease asymptotically.

6. Single Composite Body Analysis

a) Equation of motion

We now modify the previous analysis for a satellite which consists of the body B only. This satellite may either be a spin stabilized or a three-axes stabilized one. All equations through Eq. (4.3) remain valid. Since there are no contact forces and torques for a single body, equations corresponding to Eqs. (4.4) and (4.5) become

$$
\underline{\bar{A}} = [\delta_{B1}] \dot{\underline{\omega}}_B + \tilde{\omega}_B [\delta_{B2}] \underline{\omega}_B \tag{6.52}
$$

$$
\underline{P} = \underline{u}_B - [I_B] \dot{\underline{\omega}}_B + [\tilde{\dot{h}}_B] \underline{\omega}_B - \tilde{\omega}_B [I_B] \underline{\omega}_B \tag{6.53}
$$

Then the equation for $\underline{\omega}_B$, corresponding to Eq. (4.23) is

$$
\begin{aligned}
&S_{B1} \dot{\underline{\omega}}_B + S_{B3} \underline{\omega}_B + S_{B5} \tilde{\omega}_B [\delta_{B2}] \underline{\omega}_B \\
&\qquad + [\overset{*}{\delta}_B] \tilde{\omega}_B [I_B] \underline{\omega}_B - [\overset{*}{\delta}_B] \underline{u}_B = 0
\end{aligned}
\tag{6.54}
$$

b) Deterministic solutions for $\underline{\omega}_B$ and $\underline{u}_B$

It is known that in the torque-free rigid problem, a constant value of $\underline{\omega}_B$ is a solution. This constant solution vector is the nominal angular velocity vector of the satellite. With this information, a perturbation series is now assumed for $\underline{\omega}_B$. Let

$$\underline{h} = \varepsilon' \underline{x} + (\varepsilon')^2 \underline{x}_2 + (\varepsilon')^3 \underline{x}_3 + \ . \ . \ . \qquad (6.55)$$

and

$$\underline{\omega}_B = \underline{x}_0 + \underline{h}$$

where $\underline{x}_0$ is the constant nominal angular velocity vector of the satellite, and

$$\varepsilon' = |\underline{u}_B(\underline{x}_o) \quad \text{at } t = 0 \qquad (6.57)$$

Let $\underline{u}_B(\underline{x}_0+\underline{h})$ be expanded in a Taylor's series, such that

$$\overset{*}{\delta}_B \underline{u}_B = \varepsilon' \underline{w}_1 + (\varepsilon')^2 \underline{w}_2 + (\varepsilon')^3 \underline{w}_3 + \ . \ . \ . \qquad (6.58)$$

Substituting Eqs. (6.55), (6.56) and (6.58) into Eq. (6.54) and separating the coefficients of $(\varepsilon')^0$ and ε', we obtain the following equation:

$$S_{B1}\dot{\underline{x}}_1 + \left[S_{B3} + S_{B5}\tilde{x}_0\delta_{B5} - S_{B5}\widetilde{(\delta_{B2}\underline{x}_0)} + \overset{*}{\delta}_B\tilde{x}_0 I_B - \overset{*}{\delta}_B\widetilde{(I_B\underline{x}_0)}\right]\underline{x}_1$$
$$= \underline{w}_1 - \frac{1}{\varepsilon'}\left[S_{B3}\underline{x}_0 + S_{B5}\tilde{x}_0\delta_{B2}x_0 + \overset{*}{\delta}_B\tilde{x}_0 I_B x_0\right] \qquad (6.59)$$

Equation (6.59) is rewritten as

$$S_{B1}\dot{\underline{x}} + (S_{B3}+S_{B6})\underline{x}_1 = \underline{w}_1 - \frac{1}{\varepsilon}S_{B3}\underline{x}_0 - \frac{1}{\varepsilon}\left[S_{B7}(\underline{x}_0)\right]\underline{x}_0 \qquad (6.60)$$

The coefficients of $(\varepsilon')^2$ generate the equation

$$S_{B1}\dot{\underline{x}}_1 + (S_{B3}+S_{B6})\underline{x}_2 = \underline{w}_2 - \left[S_{B7}(\underline{x}_1)\right]\underline{x}_1 \tag{6.61}$$

From the coefficients of $(\varepsilon')^3$, we obtain

$$S_{B1}\dot{\underline{x}}_3 + (S_{B3}+S_{B6})\underline{x}_3 = \underline{w}_3 - \left[S_{B7}(\underline{x}_1)\right]\underline{x}_2 - \left[S_{B7}(x_2)\right]\underline{x}_1 \tag{6.62}$$

Equations from higher powers of ε' can be obtained similarly. Equations thus obtained are linear with constant coefficients and will be solved as shown in the earlier parts of this work.

Since the control torque functions $\underline{w}_i$ appear linearly in Eqs. (6.60), (6.61) etc., $\underline{w}_i$ can be shown to be a "band-bang" control by the method shown earlier. Thus the controls for a single body satellite are seen to be much simpler than those of the dual-spin configuration.

c) Random solutions for $\underline{\omega}_B$ and $\underline{u}_B$

We now follow the method outlined in Part 5 of this work to obtain the statistical parameters of the random torque responses. The functional, J, to be minimized in this case is given by

$$\begin{aligned} J = &\left[\underline{\omega}_B(0) - \overline{\underline{\omega}}_B(0)\right]^T R_0^{-1}\left[\underline{\omega}_B(0)-\overline{\underline{\omega}}_B(0)\right] \\ &+\int_0^t \Big\{ \left[z_1(t)-\overline{z}_1(t)\right]^T R_1^{-1}(t)\left[z_1(t)-\overline{z}_1(t)\right] \\ &+\left[\underline{u}_B(t)-\overline{\underline{u}}_B(t)\right]^T U_B^{-1}(t)\left[\underline{u}_B(t)-\overline{\underline{u}}_B(t)\right] \\ &+2\underline{\lambda}_1\Big[S_{B1}\dot{\underline{\omega}}_B+S_{B3}\underline{\omega}_B+S_{B5}\tilde{\omega}_B\delta_{B2}\tilde{\underline{\omega}}_B + \overset{*}{\delta}_B\tilde{\omega}_B I_B\underline{\omega}_B \\ &-\overset{*}{\delta}_B\underline{u}_B\Big] + \lambda_2\left[\underline{u}_B-\overline{\underline{u}}_B\right]\Big\} dt + 2\lambda_4(\overline{\underline{z}}_1-\underline{h}_1) \end{aligned} \tag{6.63}$$

For J to be minimum, the following set of equations are to be solved simultaneously:

$$S_{B1}\dot{\underline{\omega}}_B + S_{B3}\underline{\omega}_B + S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B + \overset{*}{\delta}_B\tilde{\omega}_B I_B\underline{\omega}_B - \overset{*}{\delta}_B\underline{u}_B = 0 \qquad (6.54)$$

$$\bar{\underline{z}}_1(t) = \underline{h}_1\left[\underline{\omega}_B(0), t\right] \qquad (6.64)$$

$$\underline{u}_B(t) = \bar{\underline{u}}_B(t) + U_B\left[\overset{*}{\delta}_B\right]^T \underline{\lambda}_1(t) \qquad (6.65)$$

$$\underline{\omega}_B(0) = \bar{\underline{\omega}}_B(0) + R_0\left[S_{B1}^T\underline{\lambda}_1(0)+\lambda_4\right] \qquad (6.66)$$

$$\underline{u}_B(t) = \bar{\underline{u}}_B(t) + U_B(t)\underline{\lambda}_2 \qquad (6.67)$$

$$z_1(t) = \bar{\underline{z}}_1(t) + R_1(t)\underline{\lambda}_4 \qquad (6.68)$$

$$S_{B1}^T\dot{\underline{\lambda}}_1 - \left[S_{B3} + \frac{\partial}{\partial\omega_B}\left(S_{B5}\tilde{\omega}_B\delta_{B2}\underline{\omega}_B + \overset{*}{\delta}_B\tilde{\omega}_B I_B\underline{\omega}_B\right)\right]^T\underline{\lambda}_1 = \left(\frac{\partial\bar{\underline{u}}_B}{\partial\underline{\omega}_B}\right)^T\underline{\lambda}_2 \qquad (6.69)$$

$$\underline{\lambda}_1(T) = 0 \qquad (6.69a)$$

These equations are reduced to

$$S_{B_1}\dot{\underline{\omega}}_B + S_{B3}\underline{\omega}_B + S_{B5}\tilde{\omega}_{B2}\underline{\omega}_B + \overset{*}{\delta}_B\tilde{\omega}_B I_B\underline{\omega}_B = \overset{*}{\delta}_B\left[\bar{\underline{u}}_B(t) + U_B(\overset{*}{\delta}_B)^T\underline{\lambda}_1(t)\right] \qquad (6.70)$$

and

$$S_{B1}^T\dot{\underline{\lambda}}_1 - \left[S_{B3} + S_{B5}\omega_B\delta_{B2} - S_{B5}\widetilde{(\delta_{B2}\underline{\omega}_B)} + \overset{*}{\delta}_B\hat{\omega}_B I_B - \overset{*}{\delta}_B\widetilde{(I_B\underline{\omega}_B)}\right]^T\underline{\lambda}_1 = 0 \qquad (6.71)$$

To solve Eqs. (6.70) and (6.71) the following series are used:

$$\underline{\omega}_B = \underline{x}_0 + \varepsilon'\cdot\underline{x}_1' + (\varepsilon')^2\underline{x}_2' + (\varepsilon')^3\underline{x}_3' + \dots \qquad (6.72)$$

$$\underline{\lambda}_1 = \underline{\lambda}_{10} + \varepsilon'\underline{\lambda}_{11} + (\varepsilon')^2\underline{\lambda}_{12} + (\varepsilon')^3\underline{\lambda}_{13} + \dots \qquad (6.73)$$

$$\overset{*}{\delta}_B\bar{\underline{u}}_B = \varepsilon'\bar{\underline{w}}_1 + (\varepsilon')^2\bar{\underline{w}} + (\varepsilon')^3\bar{\underline{w}}_3 + \dots \qquad (6.74)$$

Let U be defined by

$$\varepsilon' U = \overset{*}{\delta}_B U_B \overset{*}{\delta}_B \tag{6.75}$$

Then from Eqs. (6.70) and (6.71) the following set of equations are obtained:

$$S_{B1}^T \dot{\underline{\lambda}}_{10} - (S_{B3}+S_{B6})^T \underline{\lambda}_{10} = 0 \tag{6.76}$$

$$S_{B1}\dot{\underline{x}}_1' + (S_{B3}+S_{B6})\underline{x}_1' = \underline{\overline{w}}_1 - \frac{1}{\varepsilon} S_{B3}\underline{x}_0 - \frac{1}{\varepsilon}\left[S_{B7}(x_0)\right]\underline{x}_0 + U\underline{\lambda}_{10} \tag{6.77}$$

$$S_{B1}^T \dot{\underline{\lambda}}_{11} - (S_{B3}+S_{B6})^T \underline{\lambda}_{11} = \left[S_{B7}(\underline{x}_1')\right]\underline{\lambda}_{10} \tag{6.78}$$

$$S_{B1}\dot{\underline{x}}_2' + (S_{B3}+S_{B6})\underline{x}_2' = \underline{w}_2 - \left[S_{B7}(\underline{x}_1')\right]\underline{x}_1' + U\underline{\lambda}_{11} \tag{6.79}$$

$$S_{B1}^T \dot{\underline{\lambda}}_{12} - (S_{B3}+S_{B6})^T \underline{\lambda}_{12} = \left[S_{B7}(\underline{x}_2')\right]\underline{\lambda}_{10} + \left[S_{B7}(\underline{x}_1')\right]\underline{\lambda}_{11} \tag{6.80}$$

$$S_{B1}\dot{\underline{x}}_3' + (S_{B3}+S_{B6})\underline{x}_3' = \underline{w}_3 - \left[S_{B7}(\underline{x}_1')\right]\underline{x}_2' - \left[S_{B7}(\underline{x}_2')\right]\underline{x}_1' + U\underline{\lambda}_{12} \tag{6.81}$$

Similarly, more equations of this sequence will be obtained. It is evident that the random solutions coincide with the deterministic solutions if $U_B = 0$. These equations also will be solved by the method outlined in Part 5.

From this, the mean and the variances of the pointing error of any rigid body will be obtained as shown in Part 5.

d) Stability criteria

As in the case of the dual-spin configuration, the random parameter $\underline{\lambda}_1$ is unbounded, if the mean values of the response are maintained stable. This also implies that the variances increase indefinitely. We now obtain the criteria for stability of the mean values of $\underline{\omega}_B$.

From Eqs. (6.77), (6.79) and (6.81), it is clear that the asymptotic stability is governed by the roots of

$$S_{B1}\dot{\underline{x}}' + (S_{B3}+S_{B6})\underline{x}' = 0 \tag{6.82}$$

For a stable solution, the roots of Eq. (6.82) must have negative real parts. The characteristic equation of Eq. (6.82) is

$$\begin{aligned}&\mathrm{Det}\cdot\left[S_{B1}\right]\cdot p^3 + \mathrm{Det}\cdot\left[S_{B1}\right]\cdot \mathrm{tr}\left[S_{B1}^{-1}(S_{B3}+S_{B6})\right] p^2 \\ &+\mathrm{Det}\cdot\left[S_{B3}+S_{B6}\right]\cdot \mathrm{tr}\cdot\left[(S_{B3}+S_{B6})^{-1}S_{B1}\right]p + \mathrm{Det}\cdot\left[S_{B3}+S_{B6}\right] = 0\end{aligned} \tag{6.83}$$

Using the Routh series of Eq. (6.83), the following inequalities are obtained as the required criteria:

$$\mathrm{Det}\cdot\left[S_{B1}\right] > 0 \tag{6.84}$$

$$\mathrm{Det}\cdot\left[S_{B3}+S_{B6}\right] > 0 \tag{6.85}$$

$$\mathrm{tr}\cdot\left[S_{B1}^{-1}(S_{B3}+S_{B6})\right] > 0 \tag{6.86}$$

and

$$\mathrm{tr}\cdot\left[S_{B1}^{-1}(S_{B3}+S_{B6})\right]\cdot \mathrm{tr}\cdot\left[(S_{B3}+S_{B6})^{-1}S_{B1}\right] > 1 \tag{6.87}$$

7. Performance Criteria of Satellites

Based on the preceding analysis we now propose several criteria for evaluating the performance of satellites of different designs. It has been shown that for each satellite there is an associated outer boundary layer. The duration of this boundary layer depends on the structural properties and the nominal angular velocities of the satellites. This duration is $\sqrt{\varepsilon}$, where Eq. (3.5) defines ε as

$$\varepsilon = \text{the largest eigenvalue of } \left[A_1 A_3^{-1}\right] \tag{3.5}$$

It was shown that the boundary layer is the region of turbulent motion of the satellite caused by its flexible elements. Hence our first desirable criterion is a low value of ε.

We now consider the motions of satellites beyond the boundary layer. Clearly, all useful information can be obtained only in this quiescent zone. It is assumed that all designs to be compared are stable, so that the quiscent oscillations will eventually die out in each case. Since the rate of decay may be different for different designs, we propose that a larger marginal stability of motion be taken as the second criterion for evaluation.

Our third critical parameter is ε', which is the maximum value of external torques acting on a satellite. This parameter, ε' governs the amplitudes of vibrations which should ideally be absent.

The fourth and most important criterion that we are proposing is the average pointing error for a sensor in the interval between two consecutive measurements of the angular positions of the satellite. Let [0,T] be this interval. Then the average pointing errors E_{Ai} or E_{Bi} are defined by

$$E_{Ai} = \left\| \frac{1}{T} \int_0^t \underline{\psi}_{Ai}\, dt - \underline{x}_0 \right\| \tag{6.88}$$

and

$$E_{Bi} = \left\| \frac{1}{T} \int_0^t \underline{\psi}_{Bi}\, dt \right\| \tag{6.89}$$

where ψ_{Ai} and ψ_{Bi} are given by Eqs. (5.130) and (5.127), respectively. In the above the norm $\|\underline{v}\|$ of a vector v_i, $i = 1 - n$, is given by

$$\|\underline{v}\| = \sum_{i=1}^{n} V_i \tag{6.90}$$

8. Conclusions

We now have come to the end of our analysis. In this analysis, the method of obtaining the motion of the sensors aboard a satellite is obtained by perturbation techniques. The most probable motion is obtained after considering random torques, initial conditions and measurement errors. Several stability criteria have also been established. Finally, several performance evaluation parameters have been proposed. It is hoped that these parameters will be a useful tool in comparing the pointing accuracy of different satellites. Numerical results will be presented in the following parts of this work.