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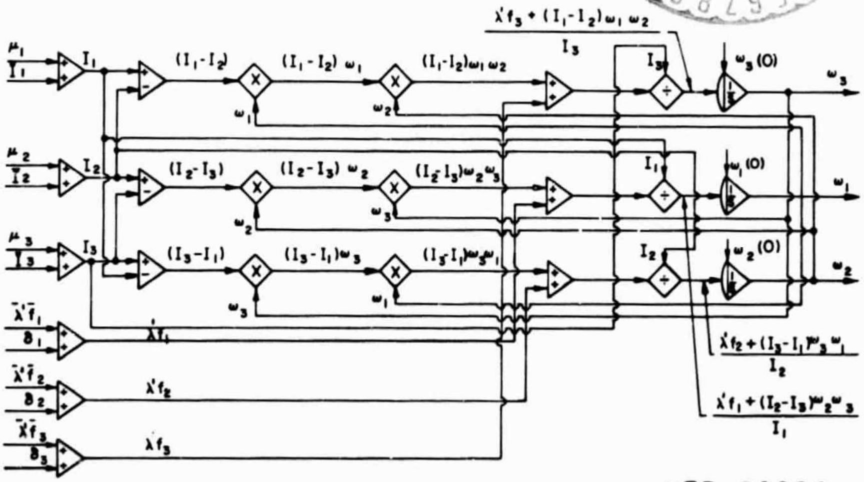

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Space Science and Engineering Center
The University of Wisconsin - Madison
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METEOROLOGICAL MEASUREMENTS

FROM SATELLITE PLATFORMS

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PREFACE

One of the great rewards enjoyed by persons who spend their lives in scientific research work is the unpredicitability of what they are doing. It is an exciting world and one I enjoy immensely. However, not all of the surprises are the kind one likes to boast about. Principal Investigators have to admit to less productive years as well as proudly present their successes.

On this program, in the meteorological study area, we have had one of those years. While good work has been done by several persons, it is as yet too incomplete to include in this report. We expect to present these efforts in next year's report.

I am happy to present three papers by Dr. Aniruddha Das and his principal advisor, Professor T. C. Huang. Publication of these papers concludes Dr. Das' development of a generalized flexible satellite attitude control model and the application of that model to some relatively simple analyses. We anticipate that Das' model will be used by government agencies and by industry in more complex applications.

I am especially grateful to Professor Huang for his assistance and support. We sincerely appreciate the patience and support of the many dedicated persons in the National Aeronautics and Space Administration with whom we have worked during the past year.

Verner E. Suomi
Principal Investigator

STABILITY OF STOCHASTIC SATELLITES

T. C. Huang and Aniruddha Das

ABSTRACT

The effects of random environmental torques and noises in the moments of inertia of spinning and three-axes stabilized satellites are compared analytically and by analog simulations. Four analytical methods are used to compute the mean values and variances of the satellite response. Among the analytical methods, it is shown that the Fokker-Planck formulation yields predictions which most coincide with the simulation results. The variances of the responses have been shown to have an initial period of growth. This growth rate falls off with time and the variances reach and stay at an equilibrium value. The growth rate is also shown to be an increasing function of the inertia noises and the nominal spin rate.

NOMENCLATURE

A_i, $i = 1\text{-}4$	= Arbitrary constants; Eq. (74).
a_i, $i = 1\text{-}27$	= Coefficients defined by Eqs. (10-18) and Eqs. (19-27).
C	= Arbitrary constant; Eq. (74).
D_1, D_2	= Arbitrary constants; Eq. (74).
$\underline{F}, \{F_i\}$	= Vector forcing function; Eqs. (91, 92).
$f^*, f^*(\underline{\omega}, t \mid \hat{\underline{\omega}}, \tau)$	= Conditional joint probability density function of $\underline{\omega}(t)$ given the values of $\hat{\underline{\omega}}(\tau)$.
$\underline{f}, \{f_i\}$; $i = 1,2,3$	= Arbitrary random forcing functions; Eqs. (1),(19)-(21).
$\overline{\underline{f}}, \{\overline{f}_i\}$; $i = 1,2,3$	= Mean values of $\underline{f}, \{f_i\}$.
G_1, G_2, G_3	= Components of $\hat{M}_{200}$, $\hat{M}_{020}$, $\hat{M}_{110}$, respectively; Eqs. (74), (74a), (74b) and (74c).
I_1, I_2, I_3	= Stochastic moments of inertia of the satellite; Eq. (1).
$\overline{I}_1, \overline{I}_2, \overline{I}_3$	= Mean values of I_1, I_2 and I_3, respectively.
J	= Functional defined by Eq. (95).
K	= Polynomial function of p; Eq. (70).
$L, L(\theta_1, \theta_2, \theta_3 \mid \hat{\omega}, t)$	= Derivative characteristic function with parameters θ_1, θ_2 and θ_3 for the random variables ω_i for a given $\hat{\omega}(t)$; Eq. (7).

L^* = Matrix differential operator; Eq. (79).

M_{ij}; i,j = 1-6 = Covariance matrix of $\underline{u}$; Eq. (5).

$\hat{M}_{klm}$ = Statistical moments of $\underline{\omega}(t)$ for a given $\hat{\underline{\omega}}(0)$; Eq. (30).

N_i, i = 1-7 = Parameters related to N_{ij} by Eq. (108).

N_{ij}; i,j = 1-6 = Covariance matrix of $\underline{v}$; Eq. (94).

p = Eigenvalue of various equations.

r = A measure of the noise levels; Eq. (122).

r_{ij}; i,j = 1-4 = Coefficients defined by Eqs. (74e) - (74g).

s_{ij}; i,j = 1-4 = Coefficients defined by Eqs. (74e) - (74g).

T = Period of time in which the most-likelihood estimates of $\underline{\omega}$ are required.

t = Time.

t_{ij}; i,j = 1-4 = Coefficients defined by Eqs. (74e) - (74g).

$\underline{u},\{u_i\}$; i = 1-6 = Random vector; Eq. (4).

$\underline{v},\{v_i\}$; i = 1-6 = Random vector; Eq. (93).

α_i, i = 1-8 = Coefficients of the characteristic polynomial for p; Eq. (41).

α_{ij}; j = 0-6 = Components of α_i; Eqs. (46), (58) etc.

β_i, i = 1-3 = Lagrangian multipliers; Eq. (95).

β_{ij}, j = 0,1,2,3,.. = Components of β_i; Eq. (109).

$\delta(t)$ = Dirac's delta function.

δ_i, i = 1-3 = White noises associated with $\lambda' f_i$; Eq. (2).

ϵ = Largest absolute value of N_{ij} for all i and j; Eq. (108a).

ϵ_i, i = 1-3 = Sample space white noises associated with I_i; Eq. (75).

η_i, i = 1-3 = Time dependent white noises associated with I_i; Eq. (75).

θ_i, i = 1-3 = Parameters of L; Eq. (7).

λf_i, i = 1-3 = Total forcing functions defined by Eqs. (10) - (12) and Eqs. (19) - (21).

$\overline{\lambda f}_i$, i = 1-3 = Mean values of λf_i.

$\lambda' f_i$, $i = 1\text{-}3$	= Total forcing functions defined by Eq. (1).
$\overline{\lambda' f_i}$, $i = 1\text{-}3$	= Mean values of $\lambda' f_i$.
λ_1, λ_2	= Parameters defined by Eqs. (71), (72).
μ_i, $i = 1\text{-}3$	= Total white noises associated with I_i; Eq. (2).
π_1	= Parameter defined by Eq. (74d).
ρ	= Parameter defined by Eq. (74d).
$\rho_{k\ell m}$	= Statistical coefficients defined by Eqs. (6), (8).
σ_i, $i = 1\text{-}3$	= Standard deviations of ω_i; Eqs. (115), (116).
Ω	= Nominal spin rate of the satellite.
$\underline{\Omega}^*, \{\Omega_i^*\}$	= Nominal angular velocity vector of the satellite.
$\underline{\omega}, \{\omega_i\}$	= Angular velocity vector of the satellite; Eq. (1).
$\hat{\underline{\omega}}, \{\hat{\omega}_i\}$	= Realized angular velocity vector corresponding to $\underline{\omega}$.
ω_{ij}, $j = 1\text{-}34$	= Components of ω_i; Eq. (76).

OPERATORS

$E\{\ \}$	= Statistical expectation.
$\overline{(\ )}$	= Mean value.
$[\ ]^T$	= Transpose.
$\dot{(\ )}$	$= \frac{d}{dt}$.

INTRODUCTION

This study compares the effects of stochastic geometry and random environmental torques on the pointing accuracy of spinning and three-axes stabilized satellites. A comparison of pointing accuracies requires a comparison of the rates of error growth over and above any criterion for the asymptotic stability of the satellites. For this reason, this study is oriented towards the determination of the statistical properties of the satellites' responses. The questions of stability have been answered indirectly by the computed responses.

The reason for considering the environmental torques on the satellites as random is self-evident. The geometries of the satellites are considered stochastic in order to have a phenomenological model of the motions of the satellites' flexible structural elements. If a satellite were absolutely rigid, its inertia properties would have been constant for all time and measured to a near certainty. Because real satellites contain many flexible and moving parts, their moments of inertia can be assumed to be stochastic variables with certain associated noise.

To be more specific, the rigid body Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3-I_2)\omega_2\omega_3 &= \lambda' f_1 \\ I_2\dot{\omega}_2 + (I_1-I_3)\omega_1\omega_3 &= \lambda' f_2 \\ I_3\dot{\omega}_3 + (I_2-I_1)\omega_1\omega_2 &= \lambda' f_3 \end{aligned} \tag{1}$$

governing the motion of satellites will now be analyzed. In the above equation, I_1, I_2, I_3 are the stochastic principal moments of inertia of the satellite. The vectors $\underline{\omega} = [\omega_1,\omega_2,\omega_3]^T$ and $\lambda'\underline{f} = [\lambda' f_1,\lambda' f_2,\lambda' f_3]^T$ are the angular velocity vector and the environmental torque vector of the satellite, respectively, along the principal axes of inertia. And λ' is a parameter. The vector $\lambda'\underline{f}$ and, consequently, the vector $\underline{\omega}$ are random variables.

Equation (1) is an example of an intrinsically nonlinear system of equations with random coefficients. The difficulty of obtaining an explicit solution to Eq. (1) can be appreciated when we realize that the stochastic version of even a simple scalar linear equation is actually nonlinear due to the dependence of the solution on the random coefficients. (See Refs. 1, 2.) The situation has been made even more complex by the presence of several contradictory methods for solving stochastic equations [1]. A widely used method of solving stochastic equations is the Fokker-Planck approach. In this, the equations are assumed to define a Markoff process and the transition probability densities of the responses are computed directly as a function of time. Several interesting equations have been solved by this method in Refs. [3-7].

Another useful method, using perturbation techniques for solving stochastic equations, was discussed in Refs. [8,9]. This is one of the "honest" methods in which response is solved analytically in terms of small random parameters. The stochastic properties of the response are obtained from the analytic solution as secondary results.

A third promising method of solution can be obtained by extending the line of logic shown in Ref. [10]. This method determines the most likelihood estimates of the response by maximizing the joint probability density of all the stochastic variables of the system. This is essentially a formulation of the Kalman filter for the case of deterministic coefficients and random forcing functions.

Lastly, there is the obvious method of initially assuming the system of equations to be deterministic and then attributing the proper stochastic properties to the deterministic solutions. It is, of course, true that this method is rigorous only if the random parameters are constants in time. The stochastic properties of the eigenvalues and eigenvectors of such systems have been computed in Refs. [11,12]. This method is worth investigating for slowly varying parameters with random step increments.

The response vector, $\underline{\omega}(t)$, of a rigid satellite governed by Eq. (1) will be analyzed using the above mentioned techniques. The analytical responses are then compared with results of an analog computer simulation. This allows verification of the relative merits of the analytic methods.

THE FOKKER-PLANCK APPROACH

This method of obtaining the response characteristics of stochastic equations is based on the analysis shown in Refs. [1,13]. The application of this method on Eq. (1) proceeds as follows:

Let the random variables μ_1, μ_2, μ_3, δ_1, δ_2 and δ_3 be defined by the equations

$$I_i = \overline{I}_i + \mu_i \; ; \; i = 1,2,3$$
$$\lambda' f_i = \overline{\lambda' f}_i + \delta_i \; ; \; i = 1,2,3 \qquad (2)$$

The bar on top of a symbol indicates mean values. Hence,

$$\overline{\mu}_i = \overline{\delta}_i = 0 \; ; \; i = 1,2,3 \qquad (3)$$

Let the stochastic vector $\underline{u}$ be defined as

$$\underline{u} = [\mu_1, \mu_2, \mu_3, \lambda' f_1, \lambda' f_2, \lambda' f_3]^T \qquad (4)$$

It is assumed that μ_i and δ_i, $i = 1\text{-}3$, are white noise disturbances, such that the matrix elements, M_{ij}, $i,j = 1\text{-}6$, are defined by

$$E\{u_i u_j\} = M_{ij}\delta(t) \qquad (5)$$

In Eq. (5) and in the following, (t) is the Dirac's delta function and the operator $E\{.\}$ denotes statistical expectation.

Let $\rho_{k\ell m}(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, t)$ be the statistical coefficients of various orders where $\hat{\omega}_i$ are the realizations of the responses ω_i, for $i = 1\text{-}3$, at any point in the time and sample spaces. Let it also be defined that $f^*[\underline{\omega}, t\,|\,\hat{\underline{\omega}}(0), 0]$ is the joint conditional probability density of the response vector, $\underline{\omega}$, given the values of $\hat{\underline{\omega}}(0)$ at $t = 0$. Thus,

$$\rho_{k\ell m} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\omega_1 - \hat{\omega}_1)^k (\omega_2 - \hat{\omega}_2)^\ell (\omega_3 - \hat{\omega}_3)^m f^*(\underline{\omega}, t+\Delta t\,|\,\hat{\underline{\omega}}, t)\, d\omega_1 d\omega_2 d\omega_3. \qquad (6)$$

Although Eq. (6) is used to define the coefficients $\rho_{k\ell m}$, these are usually calculated from the derivative characteristic function

$$L(\theta_1, \theta_2, \theta_3\,|\,\hat{\underline{\omega}}, t) \quad .$$

This, in turn, is defined by

$$L(\theta_1,\theta_2,\theta_3|\hat{\underline{\omega}},t) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} [E(\exp[i \sum_{j=1}^{3} \theta_j(\omega_j(t+\Delta t)-\omega_j(t))]|\hat{\underline{\omega}},t) - 1] \quad (7)$$

where $i = \sqrt{-1}$.

Comparing Eqs. (6) and (7), an alternative definition of $\rho_{k\ell m}$ can be obtained as

$$\rho_{k\ell m} = i^{-(k+\ell+m)} \left[\frac{\partial^{k+\ell+m} L}{\partial\theta_1^k\partial\theta_2^\ell\partial\theta_3^m}\right]\Bigg|_{\theta_1 = \theta_2 = \theta_3 = 0} \quad (8)$$

Let it be assumed that

$$\omega_j(t+\Delta t) - \omega_j(t) = \dot{\omega}_j(t)\cdot\Delta t \quad (9)$$

The values of $\rho_{k\ell m}$ are now easily calculated from Eqs. (7,8,9). For example,

$$\rho_{100} = -i\left[\frac{\partial L}{\partial\theta_1}\right]\Bigg|_{\substack{\theta_1 = 0\\ \theta_2 = 0\\ \theta_3 = 0}} = -i \lim_{\Delta t\to 0} \frac{1}{\Delta t} \frac{\partial}{\partial\theta_1} [E(\exp[i \sum_{j=1}^{3} \theta_j\omega_j\Delta t]|\hat{\underline{\omega}},t)-1]\Bigg|_{\substack{\theta_1 = 0\\ \theta_2 = 0\\ \theta_3 = 0}}$$

or $$\rho_{100} = \lim_{\Delta t\to 0} \frac{1}{\Delta t} [E(\dot{\omega}_1\Delta t|\hat{\underline{\omega}},t)]$$

or $$\rho_{100} = \lim_{\Delta t\to 0} \frac{1}{\Delta t} [E(\frac{\Delta t}{(\bar{I}_1+\mu_1)} [\lambda' f_1 - (\bar{I}_3-\bar{I}_2+\mu_3-\mu_2)\hat{\omega}_2\hat{\omega}_3]])]$$

$$\doteq \lim_{\Delta t\to 0} \frac{1}{\Delta t} E(\frac{\Delta t}{\bar{I}_1} (1 - \frac{\mu_1}{\bar{I}_1})[\lambda' f_1 - (\bar{I}_3-\bar{I}_2+\mu_3-\mu_2)\hat{\omega}_2\hat{\omega}_3])$$

Expanding the right hand side and neglecting the cubic and higher order terms in μ_j,

$$\rho_{100} = \frac{1}{\bar{I}_1} [(\frac{M_{13}-M_{12}}{\bar{I}_1} + \bar{I}_2 - \bar{I}_3)\hat{\omega}_2\hat{\omega}_3 - \frac{M_{14}}{\bar{I}_1} + \bar{\lambda}'\bar{f}_1] \quad (10)$$

Proceeding similarly, it is easily seen that

$$\rho_{010} = \frac{1}{\bar{I}_2} [(\frac{M_{12}-M_{23}}{\bar{I}_2} + \bar{I}_3 - \bar{I}_1)\hat{\omega}_1\hat{\omega}_3 - \frac{M_{25}}{\bar{I}_2} + \bar{\lambda}'\bar{f}_2] \quad (11)$$

$$\rho_{001} = \frac{1}{\bar{I}_3} [(\frac{M_{23}-M_{13}}{\bar{I}_3} + \bar{I}_1 - \bar{I}_2)\hat{\omega}_1\hat{\omega}_2 - \frac{M_{36}}{\bar{I}_3} + \bar{\lambda}'\bar{f}_3] \quad (12)$$

$$\rho_{200} = \frac{1}{\bar{I}_1^2} [\{M_{33} - 2M_{23} + M_{22} - \frac{4(\bar{I}_3-\bar{I}_2)}{\bar{I}_1} (M_{13}-M_{12})\}\hat{\omega}_2^2\hat{\omega}_3^2$$

$$+ \{\frac{4}{\bar{I}_1} (\bar{I}_3-\bar{I}_2)M_{14} - 2(M_{34}-M_{24})\}\hat{\omega}_2\hat{\omega}_3 + M_{44}] \tag{13}$$

$$\rho_{110} = \frac{1}{\bar{I}_1\bar{I}_2} [\{M_{13} + M_{23} - M_{33} - M_{12} + \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_1} (M_{13}-M_{11})$$

$$+ \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_1} (M_{13}-M_{12}) + \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_2} (M_{23}-M_{12}) + \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_2} (M_{23}-M_{22})$$

$$- \frac{(\bar{I}_3-\bar{I}_2)(\bar{I}_3-\bar{I}_1)}{\bar{I}_1\bar{I}_2} M_{12}\}\hat{\omega}_1\hat{\omega}_2\hat{\omega}_3^2 + \{M_{34} - M_{14} - \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_1} M_{14}$$

$$- \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_2} M_{24}\}\hat{\omega}_1\hat{\omega}_3 + \{M_{25} - M_{35} + \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_1} M_{15}$$

$$+ \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_2} M_{25}\}\hat{\omega}_2\hat{\omega}_3 + M_{45}] \tag{14}$$

$$\rho_{101} = \frac{1}{\bar{I}_1\bar{I}_3} [\{M_{13} - M_{23} - M_{12} + M_{22} + \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_1} (M_{11}-M_{12})$$

$$- \frac{(\bar{I}_1-\bar{I}_2)}{\bar{I}_1} (M_{13}-M_{12}) + \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_3} (M_{13}-M_{23}) - \frac{(\bar{I}_1-\bar{I}_2)}{\bar{I}_3} (M_{33}-M_{23})$$

$$- \frac{(\bar{I}_3-\bar{I}_2)(\bar{I}_1-\bar{I}_2)}{\bar{I}_1\bar{I}_3} M_{13}\}\hat{\omega}_1\hat{\omega}_2^2\hat{\omega}_3 + \{M_{14} - M_{24} - \frac{(\bar{I}_1-\bar{I}_2)}{\bar{I}_1} M_{14}$$

$$- \frac{(\bar{I}_1-\bar{I}_2)}{\bar{I}_3} M_{34}\}\hat{\omega}_1\hat{\omega}_2 + \{M_{36} - M_{26} + \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_1} M_{16}$$

$$+ \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_3} M_{36}\}\hat{\omega}_2\hat{\omega}_3 + M_{46}] \tag{15}$$

$$\rho_{020} = \frac{1}{\bar{I}_2^2} [\{M_{33} - 2M_{13} + M_{11} - \frac{4}{\bar{I}_2} (\bar{I}_3-\bar{I}_1)(M_{23}-M_{12})\}\hat{\omega}_1^2\hat{\omega}_3^2$$

$$+ \{2(M_{35}-M_{15}) - \frac{4}{\bar{I}_2} (\bar{I}_3-\bar{I}_1)M_{25}\}\hat{\omega}_1\hat{\omega}_3 + M_{55}] \tag{16}$$

$$\rho_{011} = \frac{1}{\bar{I}_2\bar{I}_3} [\{M_{13} - M_{23} - M_{11} + M_{12} + \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_2} (M_{22}-M_{12})$$

$$+ \frac{(\bar{I}_2-\bar{I}_1)}{\bar{I}_2} (M_{23}-M_{12}) + \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_3} (M_{23}-M_{13}) + \frac{(\bar{I}_2-\bar{I}_1)}{\bar{I}_3} (M_{33}-M_{13})$$

$$- \frac{(\bar{I}_3-\bar{I}_1)(\bar{I}_2-\bar{I}_1)}{\bar{I}_2\bar{I}_3} M_{23}\}\hat{\omega}_1^2\hat{\omega}_2\hat{\omega}_3 + \{M_{36} - M_{16} - \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_2} M_{26}$$

$$- \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_3} M_{36}\}\hat{\omega}_1\hat{\omega}_3 + \{M_{15} - M_{25} + \frac{(\bar{I}_2-\bar{I}_1)}{\bar{I}_2} M_{25}$$

$$+ \frac{(\bar{I}_2-\bar{I}_1)}{\bar{I}_3} M_{35}\}\hat{\omega}_1\hat{\omega}_2 + M_{56}] \tag{17}$$

$$\rho_{002} = \frac{1}{\bar{I}_3^2} [\{M_{22} - 2M_{12} + M_{11} - \frac{4}{\bar{I}_3} (\bar{I}_2-\bar{I}_1)(M_{23}-M_{13})\}\hat{\omega}_1^2\hat{\omega}_2^2$$

$$+ \{ \frac{4}{\bar{I}_3} (\bar{I}_2-\bar{I}_1)M_{36} - 2(M_{26}-M_{16})\}\hat{\omega}_1\hat{\omega}_2 + M_{66}] \tag{18}$$

All of the first and second order expressions of $\rho_{k\ell m}$ are listed in Eqs. (10-18) above. The third and higher order $\rho_{k\ell m}$ are usually small and can be neglected. Suitably defining the set of constants a_j, j = 1-27, Eqs. (10-18) can be rewritten as

$$\rho_{100} = a_1\hat{\omega}_2\hat{\omega}_3 + \bar{\lambda}\bar{f}_1 - a_2 \tag{19}$$

$$\rho_{010} = a_3\hat{\omega}_1\hat{\omega}_3 + \bar{\lambda}\bar{f}_2 - a_4 \tag{20}$$

$$\rho_{001} = a_5\hat{\omega}_1\hat{\omega}_2 + \bar{\lambda}\bar{f}_3 - a_6 \tag{21}$$

$$\rho_{200} = a_7\hat{\omega}_2^2\hat{\omega}_3^2 + a_8\hat{\omega}_2\hat{\omega}_3 + a_9 \tag{22}$$

$$\rho_{110} = a_{10}\hat{\omega}_1\hat{\omega}_2\hat{\omega}_3^2 + a_{11}\hat{\omega}_1\hat{\omega}_3 + a_{12}\hat{\omega}_2\hat{\omega}_3 + a_{13} \tag{23}$$

$$\rho_{101} = a_{14}\hat{\omega}_1\hat{\omega}_2^2\hat{\omega}_3 + a_{15}\hat{\omega}_1\hat{\omega}_2 + a_{16}\hat{\omega}_2\hat{\omega}_3 + a_{17} \tag{24}$$

$$\rho_{020} = a_{18}\hat{\omega}_1^2\hat{\omega}_3^2 + a_{19}\hat{\omega}_1\hat{\omega}_3 + a_{20} \tag{25}$$

$$\rho_{011} = a_{21}\hat{\omega}_1^2\hat{\omega}_2\hat{\omega}_3 + a_{22}\hat{\omega}_1\hat{\omega}_2 + a_{23}\hat{\omega}_1\hat{\omega}_3 + a_{24} \tag{26}$$

$$\rho_{002} = a_{25}\hat{\omega}_1^2\hat{\omega}_2^2 + a_{26}\hat{\omega}_1\hat{\omega}_2 + a_{27} \tag{27}$$

Because the values of $\rho_{k\ell m}$, corresponding to the system given by Eq. (1) are at hand, the Fokker-Planck equation involving the density $f^*[\underline{\omega},t|\underline{\hat{\omega}}(0),0]$ for that system can now be set up. This equation for the density is [1]

$$\frac{\partial f^*}{\partial t} = \sum_{k+\ell+m>0} \frac{(-1)^{k+\ell+m}}{k!\,\ell!\,m!} \frac{\partial^{k+\ell+m}}{\partial\hat{\omega}_1^k \partial\hat{\omega}_2 \partial\hat{\omega}_3^m} (\rho_{k\ell m} f^*) \quad . \tag{28}$$

Substituting Eqs. (19-27) in Eq. (28) and neglecting all third and higher order-derivatives, Eq. (28) reduces to

$$\begin{aligned}
\frac{\partial f^*}{\partial t} = {} & \frac{1}{2} [a_7\hat{\omega}_2^2\hat{\omega}_3^2 + a_8\hat{\omega}_2\hat{\omega}_3 + a_9] \frac{\partial^2 f^*}{\partial\hat{\omega}_1^2} + \frac{1}{2} [a_{18}\hat{\omega}_1^2\hat{\omega}_3^2 + a_{19}\hat{\omega}_1\hat{\omega}_3 + a_{20}] \frac{\partial^2 f^*}{\partial\hat{\omega}_2^2} \\
& + \frac{1}{2} [a_{25}\hat{\omega}_1^2\hat{\omega}_2^2 + a_{26}\hat{\omega}_1\hat{\omega}_2 + a_{27}] \frac{\partial^2 f^*}{\partial\hat{\omega}_3^2} + [a_{10}\hat{\omega}_1\hat{\omega}_2\hat{\omega}_3^2 + a_{11}\hat{\omega}_1\hat{\omega}_3 \\
& + a_{12}\hat{\omega}_2\hat{\omega}_3 + a_{13}] \frac{\partial^2 f^*}{\partial\hat{\omega}_1\partial\hat{\omega}_2} + [a_{14}\hat{\omega}_1\hat{\omega}_2^2\hat{\omega}_3 + a_{15}\hat{\omega}_1\hat{\omega}_2 + a_{16}\hat{\omega}_2\hat{\omega}_3 \\
& + a_{17}] \frac{\partial^2 f^*}{\partial\hat{\omega}_1\partial\hat{\omega}_3} + [a_{21}\hat{\omega}_1^2\hat{\omega}_2\hat{\omega}_3 + a_{22}\hat{\omega}_1\hat{\omega}_2 + a_{23}\hat{\omega}_1\hat{\omega}_3 + a_{24}] \frac{\partial^2 f^*}{\partial\hat{\omega}_2\partial\hat{\omega}_3} \\
& + [a_{10}\hat{\omega}_1\hat{\omega}_3^2 + a_{12}\hat{\omega}_3 + a_{14}\hat{\omega}_1\hat{\omega}_2^2 + a_{16}\hat{\omega}_2 - a_1\hat{\omega}_2\hat{\omega}_3 - \overline{\lambda f}_1 + a_2] \frac{\partial f^*}{\partial\hat{\omega}_1} \\
& + [a_{10}\hat{\omega}_2\hat{\omega}_3^2 + a_{11}\hat{\omega}_3 + a_{21}\hat{\omega}_1^2\hat{\omega}_2 + a_{23}\hat{\omega}_1 - a_3\hat{\omega}_1\hat{\omega}_3 - \overline{\lambda f}_2 + a_4] \frac{\partial f^*}{\partial\hat{\omega}_2} \\
& + [a_{14}\hat{\omega}_2^2\hat{\omega}_3 + a_{15}\hat{\omega}_2 + a_{21}\hat{\omega}_1^2\hat{\omega}_3 + a_{22}\hat{\omega}_1 - a_5\hat{\omega}_1\hat{\omega}_2 - \overline{\lambda f}_3 + a_6] \frac{\partial f^*}{\partial\hat{\omega}_3} \\
& + [a_{10}\hat{\omega}_3^2 + a_{14}\hat{\omega}_2^2 + a_{21}\hat{\omega}_1^2] f^* \quad .
\end{aligned} \tag{29}$$

The values of the density function can be obtained by solving this formidable linear second order partial differential equation. But little useful information is obtained from the density function. The truly useful statistical parameters are the mean values, variances, covariances, and other higher order moments of the satellite response. These parameters form a family, $\hat{M}_{k\ell m}$, which is defined by

$$\hat{M}_{k\ell m} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\omega}_1^k \hat{\omega}_2^\ell (\hat{\omega}_3 - \Omega)^m f^*[\underline{\hat{\omega}}, t \,|\, \underline{\hat{\omega}}(0), 0]\, d\hat{\omega}_1 d\hat{\omega}_2 d\hat{\omega}_3 \tag{30}$$

and hence

$$\frac{d\hat{M}_{k\ell m}}{dt} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\omega}_1^k \hat{\omega}_2^\ell (\hat{\omega}_3 - \Omega)^m \frac{\partial f^*}{\partial t}\, d\hat{\omega}_1 d\hat{\omega}_2 d\hat{\omega}_3 \ . \tag{31}$$

where Ω is the nominal value of the spin rate. Substituting the expression for $\frac{\partial f^*}{\partial t}$ from Eq. (29) into Eq. (31) and integrating, it is seen that

$$\dot{\hat{M}}_{100} = a_1 \Omega \hat{M}_{010} + a_1 \hat{M}_{011} - a_2 + \bar{\lambda}\bar{f}_1 \tag{32}$$

$$\dot{\hat{M}}_{010} = a_3 \Omega \hat{M}_{100} + a_3 \hat{M}_{101} - a_4 + \bar{\lambda}\bar{f}_2 \tag{33}$$

$$\dot{\hat{M}}_{001} = a_5 \hat{M}_{110} - a_6 + \bar{\lambda}\bar{f}_3 \tag{34}$$

$$\dot{\hat{M}}_{200} = 2(\bar{\lambda}\bar{f}_1 - a_2)\hat{M}_{100} + a_8 \Omega \hat{M}_{010} + 2a_1 \Omega \hat{M}_{110} + a_7 \Omega^2 \hat{M}_{020} + a_8 \hat{M}_{011} + a_9 \tag{35}$$

$$\dot{\hat{M}}_{110} = (a_{11}\Omega + \bar{\lambda}\bar{f}_2 - a_4)\hat{M}_{100} + (a_{12}\Omega + \bar{\lambda}\bar{f}_1 - a_2)\hat{M}_{010} + a_1 \Omega \hat{M}_{020} + a_{10}\Omega^2 \hat{M}_{110}$$
$$+ \ a_{11}\hat{M}_{101} + a_3 \Omega \hat{M}_{200} + a_{12}\hat{M}_{011} + a_{13} \tag{36}$$

$$\dot{\hat{M}}_{101} = (\bar{\lambda}\bar{f}_3 - a_6)\hat{M}_{100} + a_{16}\Omega \hat{M}_{010} + (\bar{\lambda}\bar{f}_1 - a_2)\hat{M}_{001} + a_{15}\hat{M}_{110}$$
$$+ (a_1\Omega + a_{16})\hat{M}_{011} + a_{17} \tag{37}$$

$$\dot{\hat{M}}_{020} = a_{19}\Omega \hat{M}_{100} + 2(\bar{\lambda}\bar{f}_2 - a_4)\hat{M}_{010} + a_{18}\Omega^2 \hat{M}_{200} + 2a_3 \Omega \hat{M}_{110}$$
$$+ a_{19}\hat{M}_{101} + a_{20} \tag{38}$$

$$\dot{\hat{M}}_{011} = a_{23}\Omega \hat{M}_{100} + (\bar{\lambda}\bar{f}_3 - a_6)\hat{M}_{010} + (\bar{\lambda}\bar{f}_2 - a_4)\hat{M}_{001} + a_{22}\hat{M}_{110}$$
$$+ (a_{23} + a_3\Omega)\hat{M}_{101} + a_{24} \tag{39}$$

$$\dot{\hat{M}}_{002} = 2(\bar{\lambda}\bar{f}_3 - a_6)\hat{M}_{001} + a_{26}\hat{M}_{110} + a_{27} \tag{40}$$

In deriving Eqs. (32-40), all third and higher order moments have been neglected . Solving these nine first order ordinary differential equations, the mean values, the variances, and the covariances of the satellite response are obtained completely.

THE FOKKER-PLANCK RESPONSE

At this point, it will be interesting to analyze the response predicted by Eqs. (32-40). These predictions will later be compared with an analog simulation of Eq. (1).

Let it be assumed that, at $t = 0$, all second order moments ($k+\ell+m = 2$) and $\hat{M}_{001}$ are equal to zero. In this stage, the satellite will behave as it does in the deterministic situation, that is, it will begin to precess with a rate proportional to Ω. Then, as the values of $\hat{M}_{001}$ and $\hat{M}_{002}$ grow with time, the precessing rate and the nutation angle will also grow. Finally, the satellite topples down. This phenomenon occurs physically and in simulations. Thus, Eqs. (32-40) predict that the satellite response is greatly sensitive to the values of a_5, $(\bar{\lambda}\bar{f}_3-a_6)$, a_{26} and a_{27}. Because a_{27}, a_{20}, and a_9 are non-negative, these equations predict that an uncontrolled satellite governed by Eq. (1) is inherently unstable in the presence of random errors. The same conclusion can be drawn by applying the stability criteria of Refs. [14,15] to Eq. (1). The error growth rate of the satellite response can be minimized by minimizing the values of a_5, a_6, a_{26}, and λf_3. This can be done if $\bar{\lambda}\bar{f}_3 = 0$, $\bar{I}_1 = \bar{I}_2$ and the matrix M_{ij} is a diagonal matrix.

The relative rates of error growth of spinning and non-spinning satellites will now be examined from the characteristics of the eigenvalues of Eqs. (32-40). It can be shown that the eigenvalues of these equations satisfy a ninth degree algebraic equation of the form

$$p^9 + \alpha_8 p^8 + \alpha_7 p^7 + \alpha_6 p^6 + \alpha_5 p^5 + \alpha_4 p^4 + \alpha_3 p^3 + \alpha_2 p^2 + \alpha_1 p = 0 \qquad (41)$$

where α_i, $i = 1\text{-}8$, are appropriate constants.

It is obvious that to have bounded growth rates, α_i for all i must be non-negative. It can be shown that

$$\begin{aligned}\alpha_8 = -a_{10}\Omega^2 = -\frac{\Omega^2}{\bar{I}_1\bar{I}_2}\Big[&M_{13} + M_{23} - M_{33} - M_{12} + \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_1}(M_{13}-M_{11}) \\ &+ \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_1}(M_{13}-M_{12}) + \frac{(\bar{I}_3-\bar{I}_2)}{\bar{I}_2}(M_{23}-M_{12}) + \frac{(\bar{I}_3-\bar{I}_1)}{\bar{I}_2}(M_{23}-M_{22}) \\ &\frac{(\bar{I}_3-\bar{I}_1)(\bar{I}_3-\bar{I}_2)}{\bar{I}_1\bar{I}_2}M_{12}\Big]\end{aligned} \qquad (42)$$

Because usual satellite geometries are such that

$$\overline{I}_3 > \text{Max}\ [\overline{I}_1, \overline{I}_2]$$

Eq. (42) says that $\alpha_8 \neq 0$ if, and only if any of the following conditions exist:

$$\Omega = 0 \tag{43}$$

$$\max[M_{13}, M_{23}] \leq \min\ [M_{11}, M_{12}, M_{22}, M_{33}] \tag{44}$$

In particular, $\alpha_8 \geq 0$ if $M_{12} \geq 0$ and

$$M_{13} = M_{23} = 0 \tag{45}$$

since $\min\ [M_{11}, M_{22}, M_{33}] \geq 0$.

Equation (45) states that one of the conditions for a bounded error growth rate is satisfied if the inertia noises in I_1 and I_2 are independent of the noise in I_3. But this condition usually is not satisfied because

$$I_3 = I_1 + I_2 \quad \text{and} \quad \overline{I}_3 = \overline{I}_1 + \overline{I}_2 \quad \text{and hence}$$

$$\mu_3 = \mu_1 + \mu_2$$

and, therefore,

$$M_{13} = M_{11} + M_{12}$$

$$M_{23} = M_{22} + M_{12} \quad .$$

Thus, at this point it appears that Eq. (43) provides the only suitable constraint and that this constraint is available only to three-axes stabilized satellites.

Now, let the conditions required to make α_7 non-negative be considered. It can be shown that α_7 is of the form

$$\alpha_7 = \alpha_{70} + \alpha_{71}\Omega + \alpha_{72}\Omega^2 + \alpha_{74}\Omega^4 \ , \tag{46}$$

where

$$\alpha_{70} = -\ [a_{22}a_{12} + a_{23}a_{16} + a_{15}a_{11}] \tag{47}$$

$$\alpha_{71} = -\ 2[a_1a_{23} + a_3a_{16}] \tag{48}$$

$$\alpha_{72} = -\ 6a_1a_3 \tag{49}$$

$$\alpha_{74} = -\ a_7a_{18} \tag{50}$$

Another reasonable assumption we can make now is that the inertia noises, μ_i, are independent of the forcing funcitons, λf_i. Assuming this,

$$a_2 = a_4 = a_6 = a_8 = a_{11} = a_{12} = a_{15} = a_{16} = 0 \tag{51}$$

$$a_{19} = a_{22} = a_{23} = a_{26} = 0 \tag{52}$$

Using Eqs. (51,52), the criterion for non-negative α_7 becomes either Eq. (43) or

$$\alpha_{72} + \alpha_{74}\Omega^2 \geq 0 \quad . \tag{53}$$

Equation (53) can be expanded to obtain

$$6a_1a_3 + a_7a_{18}\Omega^2 \leq 0$$

or

$$\begin{aligned} &\frac{6}{\overline{I}_1\overline{I}_2}\left[\frac{M_{13}-M_{12}}{\overline{I}_1} - (\overline{I}_3-\overline{I}_2)\right]\left[(\overline{I}_3-\overline{I}_1) + \frac{M_{12}-M_{23}}{\overline{I}_2}\right] \\ &+ \frac{\Omega^2}{\overline{I}_1^2\overline{I}_2^2}\left[M_{33} - 2M_{23} + M_{22} - \frac{4(\overline{I}_3-\overline{I}_2)}{\overline{I}_1}(M_{13}-M_{12})\right]\left[M_{33} - 2M_{12}\right. \\ &\left. + M_{11} - \frac{4(\overline{I}_3-\overline{I}_1)}{\overline{I}_2}(M_{23}-M_{12})\right] \leq 0 \end{aligned} \tag{54}$$

Assuming the satellite geometry to be given by

$$\frac{1}{2}\overline{I}_3 = \overline{I}_1 = \overline{I}_2 \tag{55}$$

$$\mu_3 = \mu_1 + \mu_2 \tag{56}$$

and that M_{ij} are small compared to $\overline{I}_1$, Eq. (54) can be further simplified to read

$$2\overline{I}_1^4 - M_{11}(3M_{22}-2M_{11})\Omega^2 \geq 0 \tag{57}$$

Equation (57) is almost certainly satisfied for all real satellites and hence, α_7 is almost certainly positive. Equation (57) also states the obvious fact that, in the presence of inertia noise, a high spin rate tends to make the satellite unstable.

The expressions for α_6 will now be considered. It can be shown that α_6 is given by

$$\alpha_6 = \alpha_{60} + \alpha_{61} + \alpha_{62}\Omega^2 + \alpha_{63}\Omega^3 + \alpha_{64}\Omega^4 + \alpha_{66}\Omega^6 \tag{58}$$

where

$$\alpha_{60} = -\,[a_{12}a_{15}a_{23} + a_{11}a_{16}a_{22} + (\lambda\bar{f}_1 - a_2)(a_3a_{15}+a_5a_{11})$$
$$+ (\lambda\bar{f}_2 - a_4)(a_1a_{22}+a_5a_{12}) + (\lambda\bar{f}_3 - a_6)(a_3a_{16}-a_1a_{23})]$$

$$\alpha_{61} = -\,[a_3a_8a_{22} + a_1a_{15}a_{19} + 2a_3a_{12}a_{15} + 2a_1a_{11}a_{22} + 4a_1a_3(\lambda\bar{f}_3 - a_6)]$$

$$\alpha_{62} = a_{10}a_{16}a_{23}$$

$$\alpha_{63} = 2[a_1a_{10}a_{23} + a_3a_{10}a_{16}]$$

$$\alpha_{64} = 2[a_1a_3a_{10} - a_3^2a_7 - a_1^2a_{18}]$$

$$\alpha_{66} = a_7a_{10}a_{18}$$

It has already been mentioned that, if $(\lambda\bar{f}_3 - a_6)$ is non-zero, then even the deterministic response is unbounded. Hence, to make any useful comparison, it must be assumed that $(\lambda\bar{f}_3 - a_6)$ is either zero or has been made so by appropriate controllers. Assuming this and the satisfaction of Eqs. (51,52), α_6 becomes

$$\alpha_6 = a_7a_{10}a_{18}\Omega^6 + 2[a_1a_3a_{10} - a_3^2a_7 - a_1^2a_{18}]\Omega^4 \ . \tag{59}$$

Hence, for non-zero values of Ω, small M_{ij}, and with the geometry given by Eqs. (55,56), the condition for non-negative values of α_6 can be obtained as

$$4\bar{I}_1^4 + 9M_{11}M_{22}\Omega^2 \geq 0 \ . \tag{60}$$

The above relation is satisfied almost certainly for all real satellites.

A similar treatment for the coefficient α_5 yields the inequality

$$\bar{I}_1^4 - 2M_{11}M_{22}\Omega^2 - \frac{\bar{I}_1^2}{36\Omega^4}\,[(\lambda\bar{f}_1)^2 - (\lambda\bar{f}_2)^2](M_{23}-M_{13}) \geq 0 \tag{61}$$

which is also satisfied.

Carrying on with this procedure, it can be shown that the coefficients α_4, α_3, α_2, and α_1 are all well behaved and positive definite. Thus, the only critical coefficient is α_8. This is approximately given by

$$\alpha_8 = -a_{10}\Omega^2 = -\frac{\Omega^2}{\bar{I}_1^2}(M_{11}+M_{22}) \tag{62}$$

where M_{11} and M_{22} are the variances of the inertia noise along I_1 and I_2, respectively.

To give a clearer picture of the error growth phenomenon, we will analyze the response of a three-axes stabilized satellite.

Let it be assumed that initially

$$\Omega = 0 \tag{63}$$

$$a_5 = 0 \tag{64}$$

$$\bar{\lambda}\bar{f}_3 - a_6 = 0 \tag{65}$$

and Eqs. (51,52) are satisfied. In this case, all coupling in Eqs. (32-40) are lost and the responses grow linearly with time, according to the relations

$$\begin{aligned}
\hat{M}_{001} &= 0 \\
\hat{M}_{100} &= [\bar{\lambda}\bar{f}_1 - a_2]t \\
\hat{M}_{010} &= [\bar{\lambda}\bar{f}_2 - a_4]t \\
\hat{M}_{200} &= [\bar{\lambda}\bar{f}_1 - a_2]^2 t^2 \\
\hat{M}_{020} &= [\bar{\lambda}\bar{f}_2 - a_4]^2 t^2 \\
\hat{M}_{002} &= 0
\end{aligned} \tag{66}$$

The growth rate of the responses is greatly changed if Eq. (65) is not used, a though Eqs. (63,64) and Eqs. (51,52) are used. In this case, the following four equations remain coupled:

$$\begin{aligned}
\dot{\hat{M}}_{100} &= a_1\hat{M}_{011} \\
\dot{\hat{M}}_{011} &= (\bar{\lambda}\bar{f}_3 - a_6)\hat{M}_{010} + (\bar{\lambda}\bar{f}_2 - a_4)\hat{M}_{001} \\
\dot{\hat{M}}_{010} &= -a_1\hat{M}_{101} \\
\dot{\hat{M}}_{101} &= (\bar{\lambda}\bar{f}_3 - a_6)\hat{M}_{100} + (\bar{\lambda}\bar{f}_1 - a_2)\hat{M}_{001}
\end{aligned} \tag{67}$$

where

$$\dot{\hat{M}}_{001} = (\bar{\lambda}\bar{f}_3 - a_6)$$

The eigenvalues of Eq. (67) satisfy the following algebraic equation:

$$[p^4 + a_1^2(\bar{\lambda}\bar{f}_3 - a_6)^2] = 0 \tag{68}$$

Equation (68) states that, apart from the linearly growing components, there will be exponential and sinusoidal components in the satellite response, when $(\bar{\lambda}\bar{f}_3-a_6)$ is large.

The above mentioned cases, identified by Eqs. (66) and (67), are extremes. A real situation can be portrayed better by assuming $(\bar{\lambda}\bar{f}_3-a_6)$ is non-zero but very small, leading to a slight coupling in Eqs. (32-40). This causes a small non-zero value of Ω to be developed, although Eqs. (51,52) are satisfied. With this compromise, the eigenvalues of Eqs. (32-40) satisfy the following characteristic equation:

$$p^2(p^2+\Omega^2)^2[p^3-a_{10}\Omega^2p^2 + (4\Omega^2-a_7a_{18}\Omega^4)p + (a_7a_{10}a_{18}\Omega^2-2a_7-2a_{18})\Omega^4]$$

$$- a_5Kp = 0 \tag{69}$$

where it is assumed that $\bar{I}_1 = \bar{I}_2 = \frac{1}{2}\bar{I}_3$ and

$$K = (\lambda_1^2-\lambda_2^2)p^4 - 8\lambda_1\lambda_2\Omega p^3 + p^2\Omega^2[(\lambda_2^2-\lambda_1^2)(5+a_7a_{18}\Omega^2)$$

$$+ 2\lambda_1\lambda_2\Omega(a_7+a_{18})] + 4p\Omega^3[\lambda_1\lambda_2(1+a_7a_{18}\Omega^2) + \Omega(\lambda_1^2a_{18}-\lambda_2^2a_7)$$

$$+ \Omega^5[a_7a_{18}\Omega(\lambda_1^2-\lambda_2^2) - 2\lambda_1\lambda_2(a_7+a_{18})] \tag{70}$$

In Eq. (70), λ_1 and λ_2 are given by

$$\lambda_1 = \bar{\lambda}\bar{f}_1 - a_2 \tag{71}$$

$$\lambda_2 = \bar{\lambda}\bar{f}_2 - a_4 \quad . \tag{72}$$

Equation (69) can be viewed with a better perspective by considering a_5, a_7, and a_{18} to be small. This reduced Eq. (69) to the form

$$p^3(p^2+\Omega^2)^2(p^2-a_{10}\Omega^2p+4\Omega^2) = 0 \tag{73}$$

It is now clear that a spinning satellite will begin to satisfy Eq. (73) immediately in the presence of noise. A three-axes stabilized satellite, on the other hand, will satisfy Eq. (73) only after a period of linear error growth. If a_{10} is equal to zero, Eq. (73) predicts a dominant cyclic response with the well known frequencies of Ω and 2Ω . The solutions of Eqs. (32-40), corresponding to the characteristic Eq. (73), are easily obtained as follows:

$$\hat{M}_{001} = \Omega$$

$$\hat{M}_{002} = \Omega^2$$

$$\hat{M}_{100} = -\frac{\lambda_2}{\Omega} + (A_1+A_3t)\cos\Omega t + (A_2-A_4t)\sin\Omega t$$

$$\hat{M}_{010} = \frac{\lambda_1}{\Omega} + (A_1+A_3t)\sin\Omega t - (A_2-A_4t)\cos\Omega t$$

$$\hat{M}_{101} = A_3\sin\Omega t + A_4\cos\Omega t$$

$$\hat{M}_{011} = A_4\sin\Omega t - A_3\cos\Omega t \tag{74}$$

$$\hat{M}_{200} = C + \exp[\tfrac{1}{2}\, a_{10}\Omega^2 t]\{D_1\cos 2\Omega t + D_2\sin 2\Omega t\} + G_1(t)$$

$$\hat{M}_{020} = C - \exp[\tfrac{1}{2}a_{10}\Omega^2 t]\{D_1\cos 2\Omega t + D_2\sin 2\Omega t\} + G_2(t)$$

$$\hat{M}_{110} = -\frac{\lambda_1\lambda_2}{\Omega^2} + \frac{\exp[\frac{1}{2}\, a_{10}\Omega^2 t]}{(16+a_{10}\Omega^2)}\{(16D_1-4a_{10}\Omega D_2)\sin 2\Omega t$$

$$- (16D_2+4a_{10}\Omega D_1)\cos 2\Omega t\} + G_3(t)$$

where A_1, A_2, A_3, A_4, C, D_1 and D_2 are arbitrary constants, and

$$G_1(t) = \frac{a_{10}\lambda_1\lambda_2}{2\Omega} - \frac{\lambda_1^2}{\Omega^2} + A_1[r_{11}\sin\Omega t + r_{12}\cos\Omega t]$$

$$+ A_2[r_{21}\sin\Omega t + r_{22}\cos\Omega t] + A_3[r_{31}t\sin\Omega t + r_{32}t\cos\Omega t$$

$$+ r_{33}\sin\Omega t + r_{34}\cos\Omega t] + A_4[r_{41}t\sin\Omega t + r_{42}t\cos\Omega t$$

$$+ r_{43}\sin\Omega t + r_{44}\cos\Omega t] \tag{74a}$$

$$G_2(t) = -\frac{a_{10}\lambda_1\lambda_2}{2\Omega} - \frac{\lambda_2^2}{\Omega^2} + A_1[s_{11}\cos\Omega t + s_{12}\sin\Omega t]$$

$$+ A_2[s_{21}\cos\Omega t + s_{22}\sin\Omega t] + A_3[s_{31}t\cos\Omega t + s_{32}t\sin\Omega t$$

$$+ s_{33}\cos\Omega t + s_{34}\sin\Omega t] + A_4[s_{41}t\cos\Omega t + s_{42}t\sin\Omega t$$

$$+ s_{43}\cos\Omega t + s_{44}\sin\Omega t] \tag{74b}$$

$$G_3(t) = A_1[t_{11}\cos\Omega t + t_{12}\sin\Omega t] + A_2[t_{21}\cos\Omega t + t_{22}\sin\Omega t]$$

$$+ A_3[t_{31}t\cos\Omega t + t_{32}t\sin\Omega t + t_{33}\cos\Omega t + t_{34}\sin\Omega t]$$

$$+ A_4[t_{41}t\cos\Omega t + t_{42}t\sin\Omega t + t_{43}\cos\Omega t + t_{44}\sin\Omega t] \tag{74c}$$

In Eqs. (74a-74c), the constants r_{ij}, s_{ij}, and t_{ij} are defined as follows:

Let π_1 and ρ be the numbers given by

$$\pi_1 = a_{10}\Omega^2$$

$$\rho = [9\Omega^2 + \pi_1^2]^{-1} \tag{74d}$$

Then,

$$t_{11} = 3\rho(3\lambda_1\Omega - \lambda_2\pi_1)$$

$$t_{12} = -3\rho(3\lambda_2\Omega - \lambda_1\pi_1)$$

$$r_{11} = \frac{2\lambda_1}{\Omega} - 2t_{11} \; ; \; r_{12} = 2t_{12} \tag{74e}$$

$$s_{11} = -\frac{2\lambda_2}{\Omega} - 2t_{12} \; ; \; s_{12} = 2t_{11}$$

$$t_{21} = 3\rho(3\lambda_2\Omega + \lambda_1\pi_1)$$

$$t_{22} = t_{11} \; ; \; r_{21} = -2t_{21} \; ; \; r_{22} = 2t_{22} - \frac{2\lambda_1}{\Omega} \tag{74f}$$

$$s_{21} = -2t_{21} \; ; \; s_{22} = 2t_{21} - \frac{2\lambda_2}{\Omega}$$

$$t_{31} = t_{11} \; ; \; t_{32} = t_{12}$$

$$t_{33} = \frac{\rho^2}{\Omega}[81\lambda_2\Omega^3 - 18\lambda_1\Omega^2\pi_1 - 21\lambda_2\Omega\pi_1^2 - 2\lambda_1\pi_1^3]$$

$$t_{34} = \frac{\rho^2}{\Omega}[81\lambda_1\Omega^3 - 72\lambda_2\Omega^2\pi_1 + 9\lambda_1\Omega\pi_1^2 + 2\lambda_2\pi_1^3]$$

$$r_{31} = \frac{2\lambda_1}{\Omega} - 2t_{31} \; ; \; r_{32} = 2t_{32} \; ; \; r_{33} = -2t_{33} - \frac{2t_{32}}{\Omega} \tag{74g}$$

$$r_{34} = 2t_{34} - \frac{2t_{31}}{\Omega} + \frac{2\lambda_1}{\Omega^2} \; ; \; s_{31} = -2t_{32} - \frac{2\lambda_2}{\Omega}$$

$$s_{32} = 2t_{31} \; ; \; s_{33} = -2t_{34} + \frac{2t_{31}}{\Omega} \; ; \; s_{34} = 2t_{33} + \frac{2t_{32}}{\Omega} + \frac{2\lambda_2}{\Omega^2}$$

$$t_{41} = -t_{21} \; ; \; t_{42} = -t_{11}$$

$$t_{43} = \frac{\rho^2}{\Omega}[81\lambda_1\Omega^3 - 72\lambda_2\Omega^2\pi_1 - 21\lambda_1\Omega\pi_1^2 + 2\lambda_2\pi_1^3]$$

$$t_{44} = \frac{\rho^2}{\Omega}\,[81\lambda_2\Omega^3 + 72\lambda_1\Omega^2\pi_1 - 21\lambda_2\Omega\pi_1^2 - 2\lambda_1\pi_1^3]$$

$$r_{41} = -2t_{41}\ ;\ r_{42} = 2t_{42} + \frac{2\lambda_1}{\Omega}\ ;\ r_{43} = -2t_{43} - \frac{2t_{42}}{\Omega} - \frac{2\lambda_1}{\Omega^2} \tag{74h}$$

$$r_{44} = 2t_{44} - \frac{2t_{41}}{\Omega}\ ;\ s_{41} = -2t_{42}\ ;\ s_{42} = 2t_{41} + \frac{2\lambda_2}{\Omega}$$

$$s_{43} = -2t_{44} + \frac{2t_{41}}{\Omega} + \frac{2\lambda_2}{\Omega^2}\ ;\ s_{44} = 2t_{43} + \frac{2t_{42}}{\Omega}$$

The nature of the functions $G_1(t)$, $G_2(t)$, and $G_3(t)$ can be given a simpler form if a_{10} is neglected in Eqs. (74a-74h). In this case, the functions are given by

$$\begin{aligned}
G_1(t) &= -\frac{\lambda_1^2}{\Omega^2} + \frac{2}{\Omega^2}\,[\lambda_1A_3 - \lambda_2\Omega(A_1+A_3t)]\cos\Omega t \\
&\quad - \frac{2}{\Omega^2}\,[\lambda_1A_4 + \lambda_2\Omega(A_2-A_4t)]\sin\Omega t \\
G_2(t) &= -\frac{\lambda_2^2}{\Omega^2} + \frac{2}{\Omega^2}\,[\lambda_2A_4 - \lambda_1\Omega(A_2-A_4t)]\cos\Omega t \\
&\quad + \frac{2}{\Omega^2}\,[\lambda_2A_3 + \lambda_1\Omega(A_1+A_3t)]\sin\Omega t \\
G_3(t) &= \frac{1}{\Omega}\,[\lambda_1(A_1+A_3t) + \lambda_2(A_2-A_4t) + \frac{1}{\Omega}\,(\lambda_2A_3+\lambda_1A_4)]\cos\Omega t \\
&\quad - \frac{1}{\Omega}\,[\lambda_2(A_1+A_3t) - \lambda_1(A_2-A_4t) - \frac{1}{\Omega}(\lambda_1A_3-\lambda_2A_4)]\sin\Omega t.
\end{aligned} \tag{74i}$$

The constants A_1, A_2, A_3, A_4, C, D_1, and D_2 are calculated from the appropriate initial conditions. Equations (66) and (74) provide a basis for comparison of the error growth rate of spinning and three axis stabilized satellites. If a_{10}, given by Eq. (62), is large and λ_1 or λ_2 are small, then a three-axes stabilized design is warranted. The reverse is also the case. Interestingly enough, all these predictions have been borne out by analog simulations.

THE PERTURBATION SCHEME

A perturbation solution of Eq. (1) will now be obtained with the assumption that $\lambda' f_i$ and that the noises associated with the moments of inertia of the satellite are small. The inertia noises are defined as

$$I_i = \bar{I}_i + \epsilon_i + \eta_i(t)\ ;\ i = 1,2,3 \tag{75}$$

where ϵ_i and η_i are the noises in the sample and time spaces, respectively. The angular velocity responses, ω_i, are assumed to be functions of the seven small parameters λ', ϵ_i and η_i of the form:

$$\begin{aligned}\omega_i = \Omega_i^* &+ \lambda'\omega_{i0} + \epsilon_1\omega_{i1} + \epsilon_2\omega_{i2} + \epsilon_3\omega_{i3} + \eta_1\omega_{i4} + \eta_2\omega_{i5} + \eta_3\omega_{i6} \\ &+ (\lambda')^2\omega_{i7} + \lambda'\epsilon_1\omega_{i8} + \lambda'\epsilon_2\omega_{i9} + \lambda'\epsilon_3\omega_{i10} + \lambda'\eta_1\omega_{i11} \\ &+ \lambda'\eta_2\omega_{i12} + \lambda'\eta_3\omega_{i13} + (\epsilon_1)^2\omega_{i14} + \epsilon_1\epsilon_2\omega_{i15} + \epsilon_1\epsilon_3\omega_{i16} \\ &+ \epsilon_1\eta_1\omega_{i17} + \epsilon_1\eta_2\omega_{i18} + \epsilon_1\eta_3\omega_{i19} + (\epsilon_2)^2\omega_{i20} + \epsilon_2\epsilon_3\omega_{i21} \\ &+ \epsilon_2\eta_1\omega_{i22} + \epsilon_2\eta_2\omega_{i23} + \epsilon_2\eta_3\omega_{i24} + (\epsilon_3)^2\omega_{i25} \\ &+ \epsilon_3\eta_1\omega_{i26} + \epsilon_3\eta_2\omega_{i27} + \epsilon_3\eta_3\omega_{i28} + (\eta_1)^2\omega_{i29} + \eta_1\eta_2\omega_{i30} \\ &+ \eta_1\eta_3\omega_{i31} + (\eta_2)^2\omega_{i32} + \eta_2\eta_3\omega_{i33} + (\eta_3)^2\omega_{i34} \ . \qquad (76)\end{aligned}$$

In Eq. (76), the cubic and higher powers of the small parameters are neglected. The quantities Ω_i^* are the nominal values of the angular velocities ω_i. It is assumed that

$$\begin{aligned}&\Omega_1^* = \Omega_2^* = 0 \\ &\Omega_3^* = \Omega = \text{a constant} \\ &\omega_1\ (0) = \omega_2(0) = 0 = [\omega_3(0) - \Omega] \qquad (77)\end{aligned}$$

Equations (75), (76), and (77) are substituted into Eqs. (1) and separate equations are then formed corresponding to each of the various combinations of the small parameters. This classical principle of separation of parameters results in only a few of the multitude of terms on the right hand side of Eq. (76) being non-zero. Thus, a more compact expansion for the angular velocities is obtained as

$$\begin{aligned}\omega_1 &= \lambda'\omega_{10} + (\lambda')^2\omega_{17} + \lambda'\epsilon_1\omega_{18} + \lambda'\epsilon_2\omega_{19} + \lambda'\epsilon_3\omega_{110} + \lambda'\eta_1\omega_{111} \\ &\quad + \lambda'\eta_2\omega_{112} + \lambda'\eta_3\omega_{113} \\ \omega_2 &= \lambda'\omega_{20} + (\lambda')^2\omega_{27} + \lambda'\epsilon_1\omega_{28} + \lambda'\epsilon_2\omega_{29} + \lambda'\epsilon_3\omega_{210} \\ &\quad + \lambda'\eta_1\omega_{211} + \lambda'\eta_2\omega_{212} + \lambda'\eta_3\omega_{213} \\ \omega_3 &= \Omega + \lambda'\omega_{30} + (\lambda')^2\omega_{37} + \lambda'\epsilon_3\omega_{310} + \lambda'\eta_3\omega_{313}\end{aligned}$$

Let L^* be a matrix differential operator defined by

$$L^*(\omega_i) = \begin{bmatrix} \bar{I}_1 & 0 & 0 \\ 0 & \bar{I}_2 & 0 \\ 0 & 0 & \bar{I}_3 \end{bmatrix} \begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} + \begin{bmatrix} 0 & (I_3-I_2)\Omega & 0 \\ (I_1-I_3)\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \qquad (79)$$

Then the perturbation equations for the components of $\underline{\omega}$ given in Eq. (78) take the form

$$L^*(\lambda'\omega_{10}) = [\ 'f_1, \lambda' f_2, \lambda' f_3]^T \qquad (79)$$

$$L^*[(\lambda')^2\omega_{17}] = [(\bar{I}_2-\bar{I}_3)(\lambda'\omega_{20})(\lambda'\omega_{30}), (\bar{I}_3-\bar{I}_1)(\lambda'\omega_{10})(\lambda'\omega_{30}),$$
$$(\bar{I}_1-\bar{I}_2)(\lambda'\omega_{10})(\lambda'\omega_{20})]^T \qquad (80)$$

$$L^*(\lambda'\varepsilon_1\omega_{18}) = \frac{1}{\bar{I}_1}[\{\lambda'\varepsilon_1\Omega(\bar{I}_3-\bar{I}_2)\omega_{20} - \lambda'\varepsilon_1 f_1\}, -\lambda'\varepsilon_1\Omega\omega_{10}\bar{I}_1, 0]^T \qquad (81)$$

$$L^*(\lambda'\varepsilon_2\omega_{19}) = \frac{1}{\bar{I}_2}[\lambda'\varepsilon_2\bar{I}_2\Omega\omega_{20}, -\lambda'\varepsilon_2\{(\bar{I}_3-\bar{I}_1)\Omega\omega_{10} + f_2\}, 0]^T \qquad (82)$$

$$L^*(\lambda'\varepsilon_3\omega_{110}) = [-\Omega\varepsilon_3\lambda'\omega_{20}, \Omega\varepsilon_3\lambda'\omega_{10}, -\varepsilon_3\lambda'\dot{\omega}_{30}]^T \qquad (83)$$

$$L^*(\lambda'\eta_1\omega_{111}) = -[\lambda'\eta_1\dot{\omega}_{10}, \lambda'\eta_1\Omega\omega_{10}, 0]^T \qquad (84)$$

$$L^*(\lambda'\eta_2\omega_{112}) = [\lambda'\eta_2\Omega\omega_{20}, -\lambda'\eta_2\dot{\omega}_{20}, 0]^T \qquad (85)$$

$$L^*(\lambda'\eta_3\omega_{113}) = [-\lambda'\eta_3\Omega\omega_{20}, \lambda'\eta_3\Omega\omega_{10}, -\lambda'\eta_3\dot{\omega}_{30}]^T \qquad (86)$$

Equations (79-86) are easily solved. In particular, assuming

$$\bar{I}_1 = \bar{I}_2 = \frac{1}{2}\bar{I}_3 \qquad (87)$$

$$\lambda' f_2 = \lambda' f_3 = 0 \quad , \qquad (88)$$

the solutions to Eq. (79) and (80) are obtained as

$$\lambda'\omega_{10} = \frac{1}{\bar{I}_1}\int_0^t \cos\Omega(t-\tau)\lambda' f_1(\tau)d\tau$$

$$\lambda'\omega_{20} = -\frac{1}{\bar{I}_1}\int_0^t \sin\Omega(t-\tau)\lambda' f_1(\tau)d\tau \qquad (89)$$

$$\omega_{30} = \omega_{17} = \omega_{27} = \omega_{37} = 0$$

and hence

$$\omega_{310} = \omega_{313} = 0 \qquad (90)$$

The perturbation solutions obtained so far from Eqs. (89,90) agree closely with the Fokker-Planck solutions given by Eq. (74). But the drawbacks of the perturbation scheme become apparent when Eqs. (81-83) are solved. Equations (81-83) predict a secular growth of the angular velocities even for the time-independent sample space inertia noises, ε_i. This is obviously not true from a physical standpoint. Thus, all perturbation equations involving ε_i, but not η_i, must be discarded and the parameters ε_i must be absorbed in $\bar{I}_i$. Equations (81-83), then, are discarded and ε_i are set equal to zero, so that Eq. (78) reduces to

$$\omega_1 = \lambda'\omega_{10} + \lambda'\eta_1\omega_{111} + \lambda'\eta_2\omega_{112} + \lambda'\eta_3\omega_{113}$$

$$\omega_2 = \lambda'\omega_{20} + \lambda'\eta_1\omega_{211} + \lambda'\eta_2\omega_{212} + \lambda'\eta_3\omega_{213}$$

$$\omega_3 = \Omega + \lambda'\eta_3\omega_{313} \quad . \tag{90a}$$

Equation (90a) predicts that, if $\lambda'f_i$ and η_i are independent, then the mean values of the amplitudes of ω_1 and ω_2 do not grow with time. It also states that the variances of the amplitudes are stable and oscillatory and that the amplitudes of oscillation of the variances are constants for all time. In other words, no growth rate of the variances of ω_i is predicted by Eq. (90a). Contrary to this prediction, it will be seen in analog simulations that the amplitudes do grow with time, even if $\lambda'f_i$ and η_i are independent.

THE MOST-LIKELIHOOD APPROACH

The method of most-likelihood estimates will now be applied to the system described by Eq. (1). As mentioned earlier, this method is based on maximizing the joint probability density of the random variables under the constraint that Eq. (1) holds. It can be shown that this method, when applied on even a linear equation, finally requires the solving of a nonlinear equation. For this reason, the nonlinear Eq. (1) needs to be linearized initially to make analytic manipulations possible.

The well-known linearized form of Eq. (1) is given by

$$\begin{aligned} I_1\dot{\omega}_1 &= F_1 \\ I_2\dot{\omega}_2 &= F_2 \\ I_3\dot{\omega}_3 &= F_3 \end{aligned} \tag{91}$$

where

$$\begin{aligned} F_1 &= \lambda'f_1 - (I_3-I_2)\Omega\omega_2 \\ F_2 &= \lambda'f_2 - (I_1-I_3)\Omega\omega_1 \\ F_3 &= \lambda'f_3 \end{aligned} \tag{92}$$

Let $\underline{v}$ be the vector defined by

$$\underline{v} = [\mu_1,\mu_2,\mu_3,\delta_1,\delta_2,\delta_3]^T \tag{93}$$

Let the matrix elements N_{ij} be defined by

$$E\,\{v_i\ v_j\} = N_{ij}\delta(t) \tag{94}$$

Let the functional J be defined by

$$J = \sum_{i,j,k} \int_0^T \{v_i[N^{-1}]_{ij}\ v_j + 2\beta_k[I_k\dot{\omega}_k - F_k]\}dt \tag{95}$$

where β_i are arbitrary time-dependent Lagrangian multipliers. It can be shown [10] that the most likelihood estimates of ω_i are obtained by minimizing the functional J in the interval [0,T] with respect to the variables v_i and ω_i.

The variational equations for minimizing J are given by Eq. (91) and the following two equations:

$$\sum_j [N^{-1}]_{ij}\ v_j + \frac{\partial}{\partial v_i}\sum_k \beta_k(I_k\dot{\omega}_k - F_k) = 0 \tag{96}$$

$$\frac{d}{dt}[I_k\beta_k] + \sum_j \beta_j\frac{\partial F_j}{\partial\omega_k} = 0 \tag{97}$$

The terminal point condition on β is given by

$$\beta_i(T) = 0. \tag{98}$$

Assuming that

$$N_{13} = N_{23} = N_{31} = N_{32} = 0 \tag{99}$$

and

$$\begin{aligned} N_{4j} &= N_{j4} = 0 \quad \text{if } j \neq 4 \\ N_{5j} &= N_{j5} = 0 \quad \text{if } j \neq 5 \\ N_{6j} &= N_{j6} = 0 \quad \text{if } j \neq 6 \end{aligned} \tag{100}$$

Equation (96) can be opened up to read

$$\begin{aligned} \mu_1 &= N_{11}\ [\beta_1\dot{\omega}_1 + \Omega\omega_1\beta_2] + N_{12}\ [\beta_2\dot{\omega}_2 - \Omega\beta_1\omega_2] \\ \mu_2 &= N_{12}\ [\beta_1\dot{\omega}_1 + \Omega\omega_1\beta_2] + N_{22}\ [\beta_2\dot{\omega}_2 - \Omega\beta_1\omega_2] \\ \mu_3 &= N_{33}\ [\beta_3\dot{\omega}_3 + \Omega(\beta_1\omega_2 - \omega_1\beta_2)] \\ \delta_1 &= -N_{44}\beta_1 \end{aligned} \tag{101}$$

$$\delta_2 = -N_{55}\beta_2$$

$$\delta_3 = -N_{66}\beta_3$$

Using Eqs. (92) and (101), Eqs. (91) and (97) can now be reduced to the following forms:

$$[\bar{I}_1 + N_{11}(\beta_1\dot{\omega}_1+\Omega\omega_1\beta_2) + N_{12}(\beta_2\dot{\omega}_2-\Omega\beta_1\omega_2)]\dot{\omega}_1 + \Omega[\bar{I}_3-\bar{I}_2$$
$$+ N_{33}\{\beta_3\dot{\omega}_3 + \Omega(\beta_1\omega_2-\omega_1\beta_2)\} - N_{12}(\beta_1\dot{\omega}_1+\Omega\omega_1\beta_2)$$
$$- N_{22}(\beta_2\dot{\omega}_2-\Omega\beta_1\omega_2)]\omega_2 - \bar{\lambda}'\bar{f}_1 + N_{44}\beta_1 = 0 \qquad (102)$$

$$[\bar{I}_2 + N_{12}(\beta_1\dot{\omega}_1 + \Omega\omega_1\beta_2) + N_{22}(\beta_2\dot{\omega}_2-\Omega\beta_1\omega_2)\dot{\omega}_2 + \Omega[\bar{I}_1-\bar{I}_3$$
$$- N_{11}(\beta_1\dot{\omega}_1+\Omega\omega_1\beta_2) + N_{12}(\beta_2\omega_2-\Omega\beta_1\omega_2) - N_{33}\{\beta_3\dot{\omega}_3$$
$$+ \Omega(\beta_1\omega_2-\omega_1\beta_2)\}]\omega_2 - \bar{\lambda}'\bar{f}_2 + N_{55}\beta_2 = 0 \qquad (103)$$

$$[\bar{I}_3 + N_{33}\{\beta_3\dot{\omega}_3 + \Omega(\beta_1\omega_2-\omega_1\beta_2)\}]\dot{\omega}_3 - \bar{\lambda}'\bar{f}_3 + N_{66}\beta_3 = 0 \qquad (104)$$

$$[\bar{I}_1 + N_{11}(\beta_1\dot{\omega}_1+\Omega\omega_1\beta_2) + N_{12}(\beta_2\dot{\omega}_2-\Omega\beta_1\omega_2)]\dot{\beta}_1 - \Omega[\bar{I}_1-\bar{I}_3$$
$$+ N_{11}(\beta_1\dot{\omega}_1+\Omega\omega_1\beta_2) + N_{12}(\beta_2\dot{\omega}_2-\Omega\beta_1\omega_2) - N_{33}\{\beta_3\dot{\omega}_3$$
$$+ \Omega(\beta_1\omega_2-\omega_1\beta_2)\}]\beta_2 = 0 \qquad (105)$$

$$[\bar{I}_2 + N_{12}(\beta_1\dot{\omega}_1+\Omega\omega_1\beta_2) + N_{22}(\beta_2\dot{\omega}_2-\Omega\beta_1\omega_2)]\dot{\beta}_2 - \Omega[\bar{I}_3-\bar{I}_2$$
$$+ N_{33}\{\beta_3\dot{\omega}_3 + \Omega(\beta_1\omega_2-\omega_1\beta_2)\} - N_{12}(\beta_1\dot{\omega}_1+\Omega\omega_1\beta_2)$$
$$- N_{22}(\beta_2\dot{\omega}_2-\Omega\beta_1\omega_2)]\beta_1 = 0 \qquad (106)$$

$$[\bar{I}_3 + N_{33}\{\beta_3\dot{\omega}_3 + \Omega(\beta_1\omega_2-\omega_1\beta_2)\}]\dot{\beta}_3 = 0 \qquad (107)$$

Equations (102-107), together with the initial conditions on ω_i and the end conditions on β_i given by Eq. (98), form the final two-point boundary value problem coverning the stochastic motion of the satellite. To solve this problem, a perturbation sequence for β_i and ω_i has to be adopted.

Let it be assumed that ε is a small parameter and the numbers N_{ij} are of the order of ε or less. Let N_i, i = 1-7, be defined as

$$N_{11} = \varepsilon N_1$$

$$N_{12} = \varepsilon N_2$$

$$
\begin{aligned}
N_{22} &= \varepsilon N_3 \\
N_{33} &= \varepsilon N_4 \\
N_{44} &= \varepsilon N_5 \\
N_{55} &= \varepsilon N_6 \\
N_{66} &= \varepsilon N_7
\end{aligned}
\tag{108}
$$

where

$$
\varepsilon = \max_{i,j} |N_{ij}| \quad . \tag{108a}
$$

Let the variables ω_i and β_i be assumed in the form

$$
\begin{aligned}
\omega_i &= \omega_{i0} + \varepsilon\omega_{i1} + \varepsilon^2\omega_{i2} + \ldots \\
\beta_i &= \beta_{i0} + \varepsilon\beta_{i1} + \varepsilon^2\beta_{i2} + \ldots
\end{aligned}
\tag{109}
$$

such that

$$
\beta_{ij}(T) = 0 \tag{110}
$$

Substituting Eqs. (108) and (109) in Eqs. (102-107) and separating the coefficients of ε^0, ε^1, ε^2, etc., it can be seen that the zeroth order response is given by

$$
\beta_{10} = 0 \tag{111}
$$

$$
\begin{aligned}
\bar{I}_1\dot{\omega}_{10} + (\bar{I}_3-\bar{I}_2)\Omega\omega_{20} &= \overline{\lambda' f}_1 \\
\bar{I}_2\dot{\omega}_{20} - (\bar{I}_3-I_1)\Omega\omega_{10} &= \overline{\lambda' f}_2 \\
\bar{I}_3\dot{\omega}_{30} &= \overline{\lambda' f}_3
\end{aligned}
\tag{112}
$$

After some involved algebra and the use of Eq. (110), it can be seen that the predicted response from the higher order perturbation equations has essentially the same characteristics as that obtained by the straight forward perturbation scheme explained in the preceding section. Thus, the method of the most likelihood estimates suffers from the same drawbacks as those of the perturbation method.

THE METHOD OF STOCHASTIC EIGENVALUES

According to this method, the deterministic solutions of Eq. (1) are to be obtained first. Stochasticity is then imposed on these solutions to estimate the behavior of the system which was random from the beginning. Although this method is not exact, it is much simpler than the methods previously discussed.

For example, the approximate deterministic response of a three-axes stabilized satellite is given by

$$\omega_i = \int_0^t \frac{\lambda' f_i}{I_i} dt \quad , \; i = 1,2,3 \tag{113}$$

Hence, assuming $\overline{\lambda' f}_i$ to be a constant, the mean values and the variances of ω_i are given by [16]

$$E\{\omega_i\} = \frac{\overline{\lambda' f}_i}{\overline{I}_i} t \tag{114}$$

$$E\{\omega_i^2\} = (\sigma_i)^2 t^2 \tag{115}$$

where

$$\sigma_1^2 = \frac{1}{\overline{I}_1^2} \left[\frac{(\overline{\lambda' f}_1)^2 M_{11} + \overline{I}_1^2 M_{44}}{\overline{I}_1^2 + M_{11}}\right]$$

$$\sigma_2^2 = \frac{1}{\overline{I}_2^2} \left[\frac{(\lambda' f_2)^2 M_{22} + \overline{I}_2^2 M_{55}}{\overline{I}_2^2 + M_{22}}\right] \tag{116}$$

$$\sigma_3^2 = \frac{1}{\overline{I}_3^2} \left[\frac{(\overline{\lambda' f}_3)^2 M_{33} + \overline{I}_3^2 M_{66}}{\overline{I}_3^2 + M_{33}}\right]$$

In deriving Eq. (116), it was assumed that μ_i and δ_i are Gaussian random variables.

For the case of a spinning satellite with $\overline{I}_1 = \overline{I}_2 = \frac{1}{2}\overline{I}_3$, $\overline{\lambda' f}_3 = 0$, and constant values of $\overline{\lambda' f}_1$ and $\overline{\lambda' f}_2$, the deterministic amplitudes and frequency of oscillation of ω_1 and ω_2 are given by

$$\text{Freq.}[\omega_1] = \text{Freq.}[\omega_2] = \omega_3$$

$$\text{Amp.}[\omega_1] = \frac{\lambda' f_2}{\overline{I}_2 \omega_3} \simeq \frac{\lambda' f_2}{\overline{I}_2 \Omega} \tag{117}$$

$$\text{Amp.}[\omega_2] = \frac{\lambda' f_1}{\overline{I}_1 \omega_3} = \frac{\lambda' f_1}{\overline{I}_1 \Omega}$$

when $\omega_1(0) = \omega_2(0) = 0$.

Hence, the growth rates of the amplitudes and frequency are described by the variances, which are

$$E\{[\text{Freq.}[\omega_1]]^2\} = E\{[\text{Freq.}[\omega_2]]^2\} = (\sigma_3)^2 t^2 \tag{118}$$

$$E\{[\text{Amp.}[\omega_1]]^2\} = \frac{1}{\bar{I}_2^2 \Omega^2} \left[\frac{(\bar{\lambda}'\bar{f}_2)^2 \sigma_3^2 t^2 + \Omega^2 M_{55}}{\Omega^2 + \sigma_3^2 t^2}\right]$$

$$E\{[\text{Amp.}[\omega_2]]^2\} = \frac{1}{\bar{I}_1^2 \Omega^2} \left[\frac{(\bar{\lambda}'\bar{f}_1)^2 \sigma_3^2 t^2 + \Omega^2 M_{44}}{\Omega^2 + \sigma_3^2 t^2}\right] \tag{119}$$

From Eqs. (114) and (115), it is seen that the approximate predictions for the responses of three-axes stabilized satellites are quite satisfactory. Equation (118) approximately predicts the frequency growth phenomenon. Equation (119) predicts that, when t is small, such that $\sigma_3 t$ is small compared to Ω, the variances are of the form

$$E\{[\text{Amp.}[\omega_1]]^2\} = \frac{1}{\bar{I}_2^2 \Omega^2} \left[\frac{(\lambda' f_2)^2 \sigma_3^2 t^2}{\Omega^2} + M_{55}\right] \tag{120}$$

But for large values of t, the variances will reach a constant value. This is given by

$$E\{[\text{Amp.}[\omega_1]]^2\} = \left(\frac{\bar{\lambda}'\bar{f}_2}{I_2 \Omega}\right)^2 . \tag{121}$$

The prediction of an initially growing variance finally levelling off to a constant value is satisfactory and is corroborated by analog simulations. The only problem with Eqs. (120) and (121) is that these equations predict a lower growth rate and a lower value of the asymptotic variance as Ω becomes large. In this respect, Eqs. (120) and (121) differ from the Fokker-Planck formulation and the analog simulations which give higher growth rates and higher values of the asymptotic variance for larger values of Ω.

ANALOG SIMULATION

The results of simulation of the satellite response, as given by Eq. (1), can now be presented. The simplified system block diagram is shown in Figure 1. This system was programmed on an AD-256 (Analytical Dynamics-256) analog computer. The white noise inputs μ_i and δ_i, i = 1-3, were obtained from a coupled SDS-930 (Scientific Data Systems-930) real time digital computer. A high frequency RO (Repetitive Operation) clock circuit from the AD-256 was used to trigger a pseudo-random number generating program in the SDS-930. Samples of twenty such pseudo-random numbers were used to form a Gaussian white noise sequence with a zero mean value and suitable peak values. Six such independent noise sequences were continuously generated in the SDS-930 and fed to the AD-256 through six DAC (Digital to Analog Converter) lines. One test line was also used to interrupt the SDS-930 and change the peak values of the noise sequences. A sample of the noise sequences μ_i, $i = 1\text{-}3$, is shown in Figure 2 at a high brush recorder speed. At any instant of time, the frequencies of generation and the peak values and, hence, the bandwidth of all μ_i and δ_i $i = 1\text{-}3$, were maintained equal. Thus, δ_i, $i = 1\text{-}3$, are similar in nature to that shown in Figure 2, although all six noise sequences were independent of each other.

Let r be the ratio defined by

$$r = [\text{Peak value of } \mu_i \text{ and } \delta_i,\ i = 1\text{-}3]/\overline{I}_3 \qquad (122)$$

where $\overline{I}_3$ is the nominal moment of inertia about the spin-axis. Brush records of the simulated angular velocities ω_1, ω_2, and ω_3, for different values of r and Ω, are shown in Figures 3-15. The values of r and Ω, corresponding to each of these figures, are tabulated in Table 1. In all cases the initial values of ω_1 and ω_2 were taken to be zero.

Table 1: Index to the attached figures showing samples of the stochastic satellite responses.

Values of r / Values of Ω, rad./sec.	$r = \frac{1}{12} = 0.083$	$r = \frac{1}{6} = 0.166$	$r = \frac{1}{4} = 0.25$
	Figure Nos.	Figure Nos.	Figure Nos.
Fast spinner: $\Omega = 1.0$	3,4	5	6,7
Slow spinner: $\Omega = 0.5$	8	9	10,11
Three-axes stabilized: $\Omega = 0.0$	12	13	14,15

EVALUATIONS AND COMPARISON

The results of the analog simulation will now be evaluated and compared with the predictions of the analytical methods discussed earlier.

The first important result of the simulation study is that, in every case, the responses grow with time. The growth phenomenon is predicted by all four of the analytic methods only for the case of a three-axes stabilized satellite. This was true because, if $\Omega = 0$, Eq. (1) leads to a perturbed equation given by

$$I_i\dot{\omega}_i = \lambda' f_i \; ; \; i = 1,2,3 \tag{123}$$

Responses given by the solutions of Eq. (123) are the integrals of $\lambda' f_i$ and, hence, must grow linearly with time, even if $\overline{\lambda' f_i}$ are equal to zero. But, in the case of spinning satellites, only the Fokker-Planck formulation predicts an initial exponential growth. The perturbation method and the most-likelihood approach predict a constant variance. The stochastic eigenvalue method also predicts a linear growth rate which, however, is inversely proportional to Ω^2. Looking at Figures 4, 8, and 12, or at Figures 5, 9, and 13, or at Figures 7, 11 and 15, it is seen that the variances increase with Ω. Thus, at this point, the Fokker-Planck formulation is apparently the best of the theories under consideration.

A second interesting result, discernible from Figures 3, 7 and 11, is that, with time, the response amplitudes reach a stable value. Such stable values are predicted directly by the stochastic eigenvalue method. The perturbation method and the most-likelihood approach also yield the same result if it is assumed that these methods are valid only for the asymptotic case. It is to be noted that the Fokker-Planck formulation can also be made to yield this result, although not as directly as the other methods. To do this, let the solutions of $\hat{M}_{200}$ and $\hat{M}_{020}$ as given by Eq. (74) be considered:

$$\begin{aligned} \hat{M}_{200} &= C + \exp[\tfrac{1}{2} a_{10}\Omega^2 t]\{D_1\cos 2\Omega t + D_2 \sin 2\Omega t\} + G_1(t) \\ \hat{M}_{020} &= C - \exp[\tfrac{1}{2} a_{10}\Omega^2 t]\{D_1\cos 2\Omega t + D_2 \sin 2\Omega t\} + G_2(t) \end{aligned} \tag{124}$$

The exponential terms in $\hat{M}_{200}$ and $\hat{M}_{020}$ appear with opposite signs.

According to Eq. (124), one of the variances must grow and the other decay with time. Thus after a certain time, one of these variances will tend to be negative. But variances are by definition non-negative quantities. Hence, D_1 and D_2 are to be taken as non-zero until one of the variances first becomes zero. D_1 and D_2 should then be set equal to zero in order not to have negative values of $\hat{M}_{200}$ and $\hat{M}_{020}$. This procedure yields the prediction that the response amplitudes become stable after a certain time, which is in agreement with the simulation results.

The last obvious result obtained from the simulation is that, for a given value of Ω, the variances and the growth rates increase with r. This is expected, both intuitively and rationally, and all four theories predict it.

A comparison can now be made of the theoretical methods, based on purely analytical grounds. The strength of the Fokker-Planck method lies in the fact that it does not require either uncoupling or linearization of coupled nonlinear systems such as that of Eq. (1). The statistical moments of all orders are obtained directly as the solution of a coupled linear set of equations. Hence, digital computer methods can be used easily to solve such equations. The other three methods are based on initial linearization and possible uncoupling. This linearization results in a loss of useful statistical information.

There are, however, some disadvantages of the Fokker-Planck method. The primary disadvantage is that all statistical moments are coupled. Hence, when the number of dependent variables is large, the resulting set of equations is more so, even if the third and higher order moments are neglected. This method then requires some foreknowledge of the higher order moments and the statistical forms of the input random functions.

In view of the above discussion, the following conclusions can be made:

i) The Fokker-Planck formulation yields the most complete information on the responses of a satellite with random disturbing torques and stochastic moments of inertia.

ii) For a satellite with very small inertia noises, the spinning configuration is better than a three-axes stabilized configuration. The reverse is also the case.

iii) In all cases, the responses have an initial fast rate of growth. But after some time, this growth rate falls off, leading to a constant variance level depending on the variances of the input disturbing torque and on the mean moments of inertia of the satellite.

ACKNOWLEDGEMENTS

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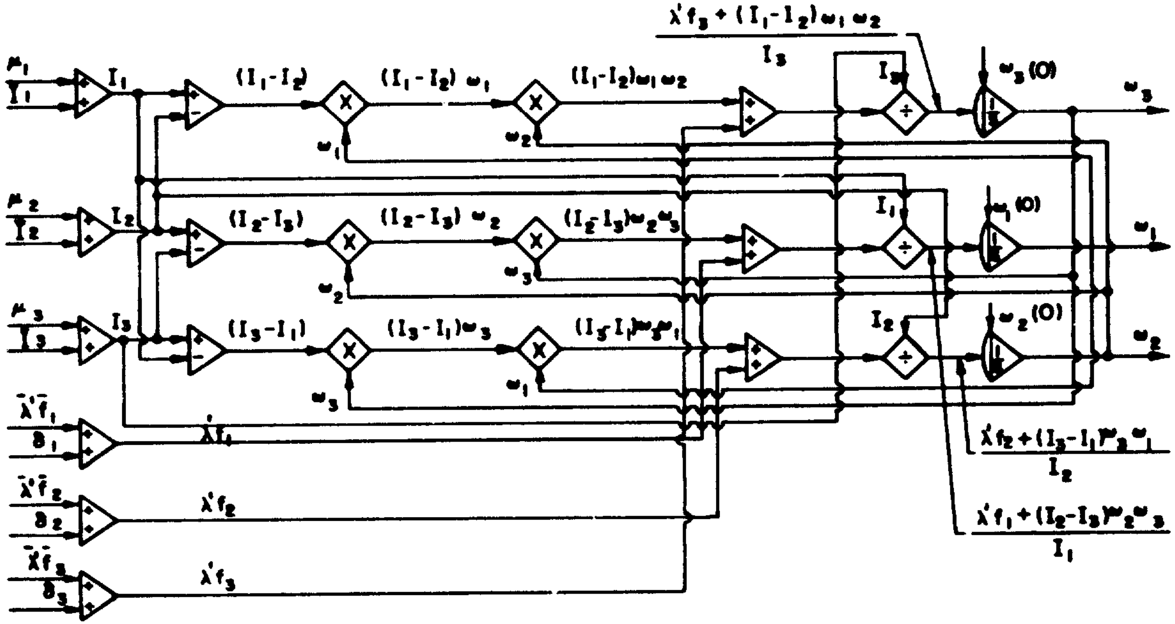


FIGURE 1. SIMULATION BLOCK DIAGRAM.

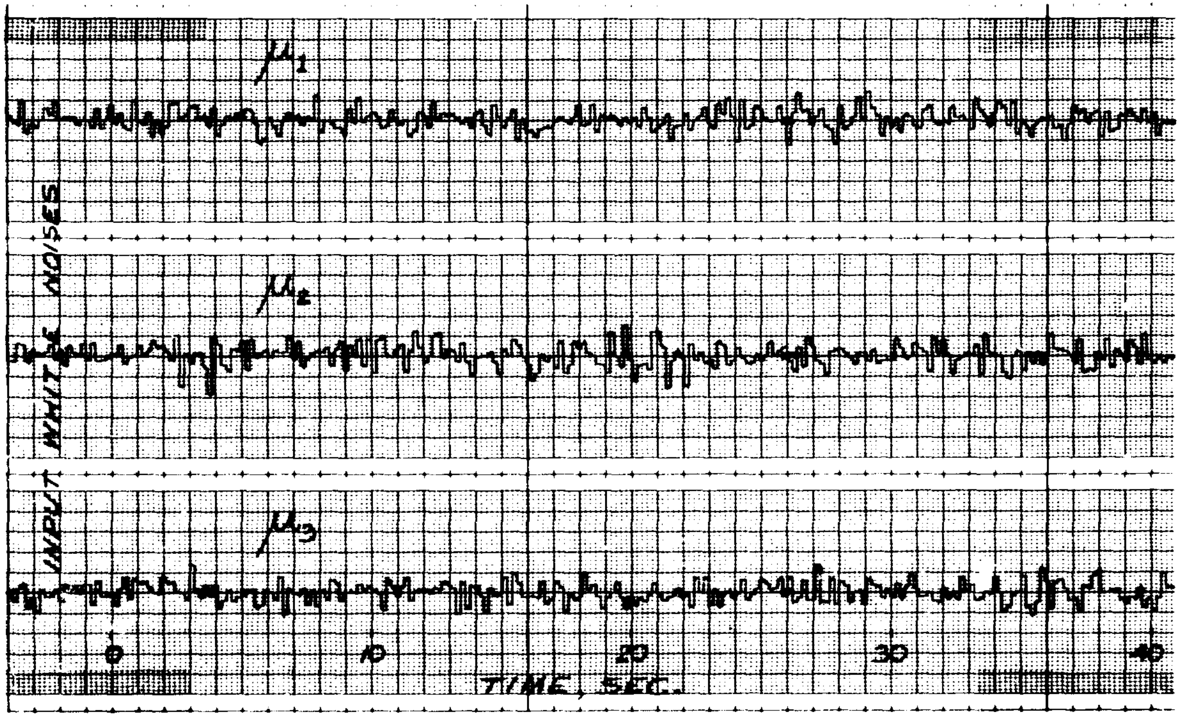
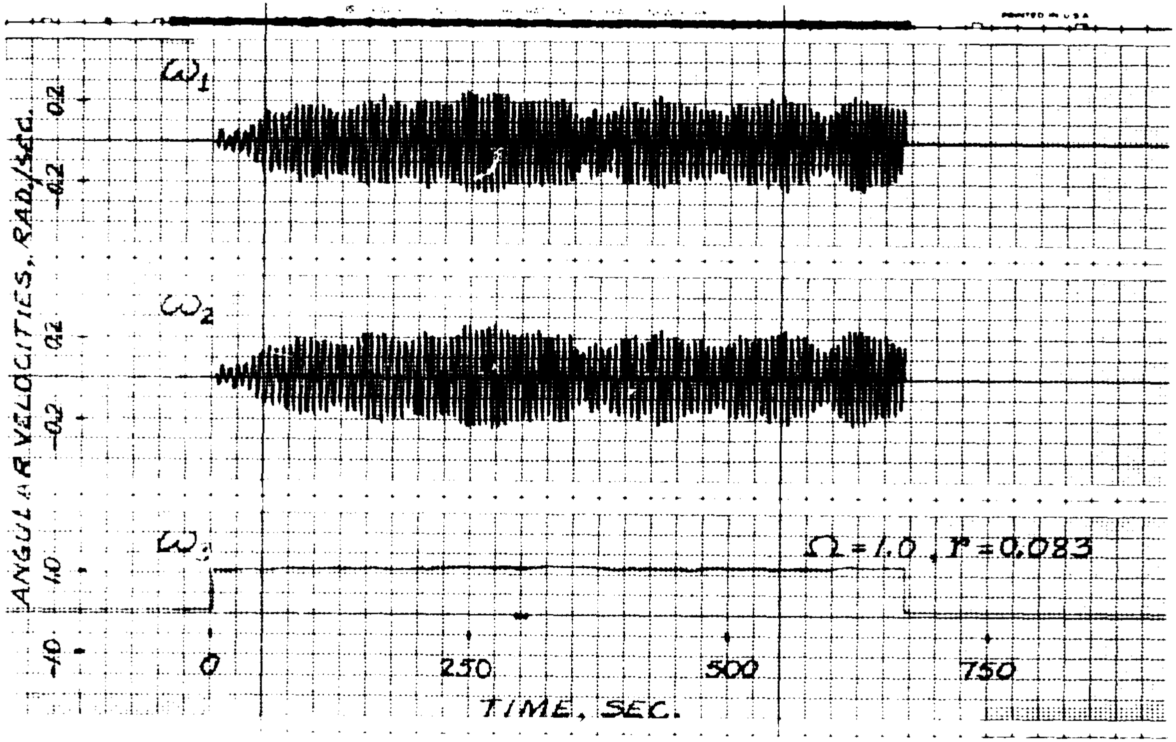


Figure 2: Input white noises.



ANGULAR VELOCITIES, RAD./SEC.
ω_1
0.2
-0.2
ω_2
0.2
-0.2
ω_3
1.0
-1.0
$\Omega = 1.0$, $\Gamma = 0.083$
0
250
500
750
TIME, SEC.

Figure 3.

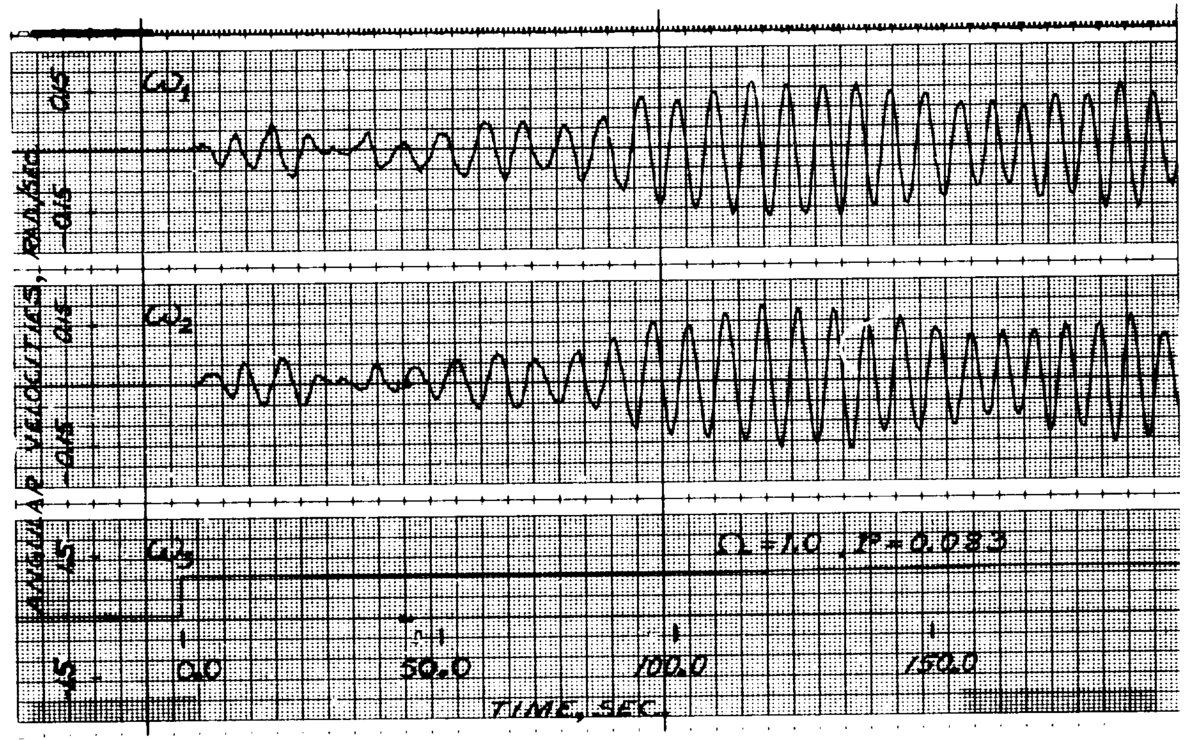

ANGULAR VELOCITIES, RAD/SEC
ω1
ω2
ω3
0.15
-0.15
.15
-.15
Ω = 1.0 , P = 0.083
0.0
50.0
100.0
150.0
TIME, SEC

Figure 4.

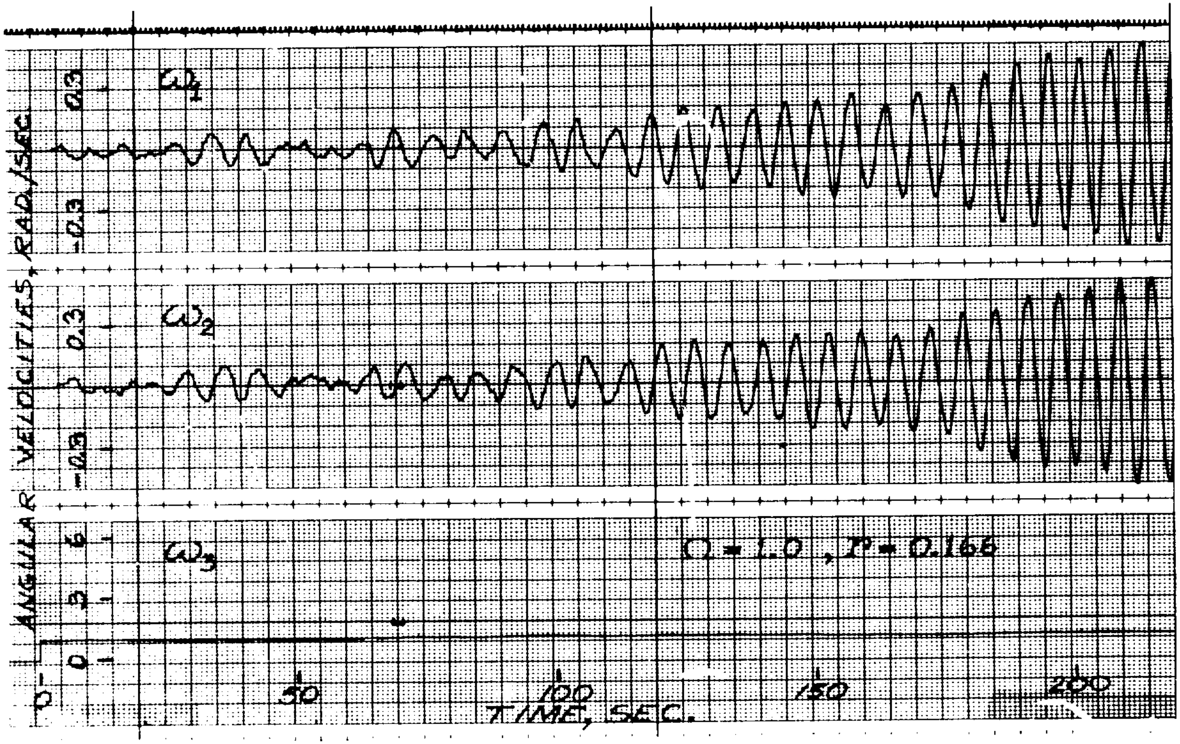

ANGULAR VELOCITIES, RAD/SEC
0.3
-0.3
ω1
ω2
ω3
6
3
0
50
100
150
200
TIME, SEC.

Figure 5.

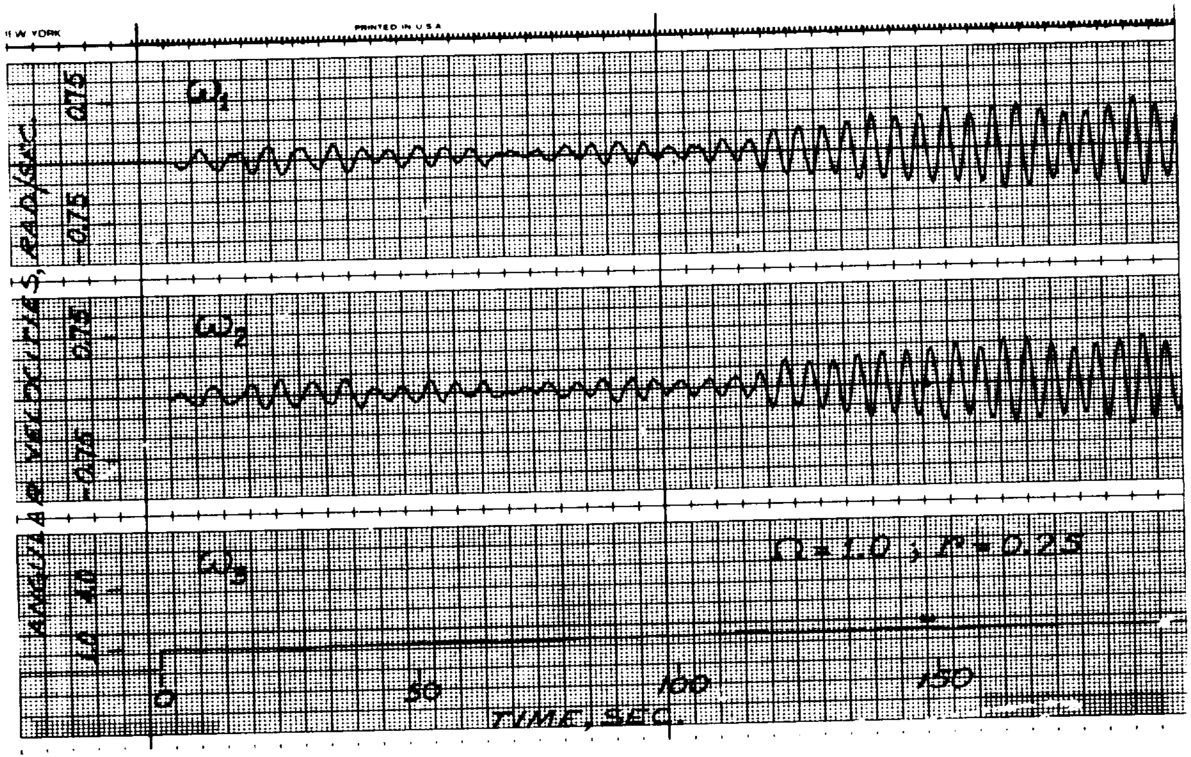
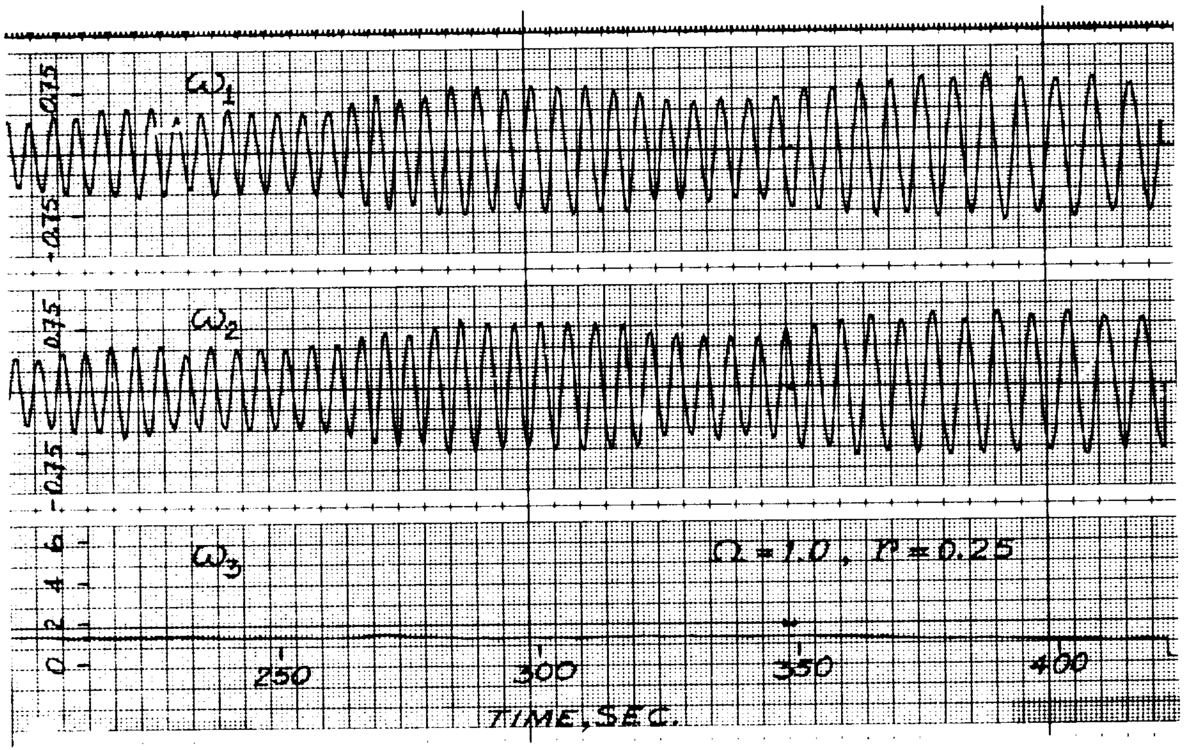
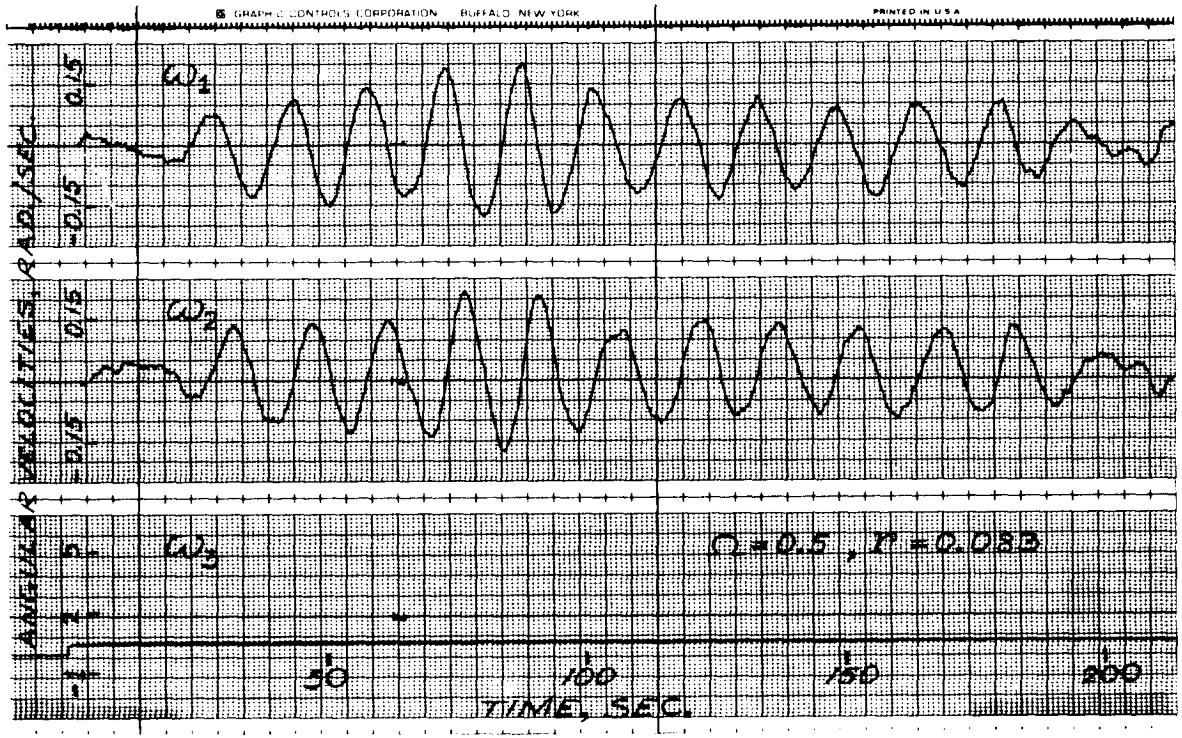

ANGULAR VELOCITIES, RAD/SEC
0.75
-0.75
ω_1
0.75
-0.75
ω_2
ω_3
1.0
0
50
100
150
TIME, SEC.

Figure 6.



ω_1
0.75
-0.75
ω_2
0.75
-0.75
ω_3
6
4
2
0
250
300
350
400
TIME, SEC.
$\Omega = 1.0,\ P = 0.25$

Figure 7.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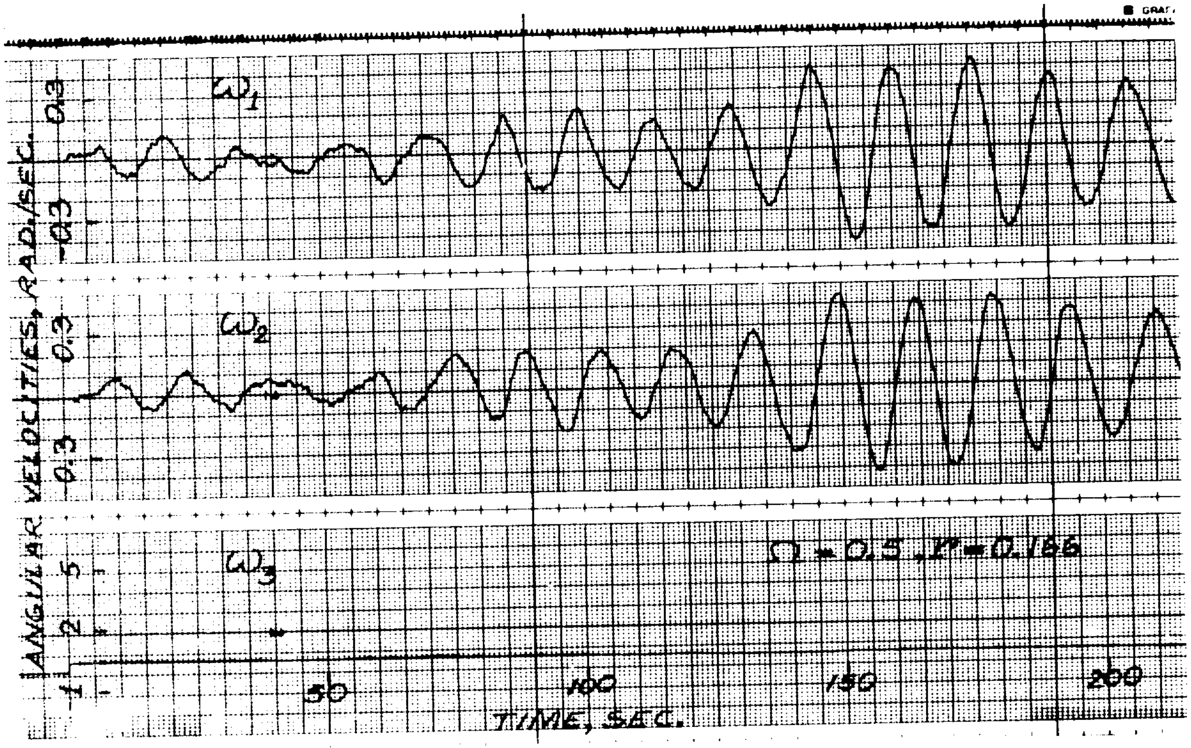
PRINTED IN U.S.A.
ANGULAR VELOCITIES, RAD/SEC
ω_1
0.15
-0.15
ω_2
0.15
-0.15
ω_3
5
$\Omega = 0.5$, $\Gamma = 0.083$
50
100
150
200
TIME, SEC.

Figure 8.



ANGULAR VELOCITIES, RAD./SEC.
ω_1
0.3
-0.3
ω_2
0.3
-0.3
ω_3
5
2
1
$\Omega = 0.5, P = 0.166$
50
100
150
200
TIME, SEC.

Figure 9.

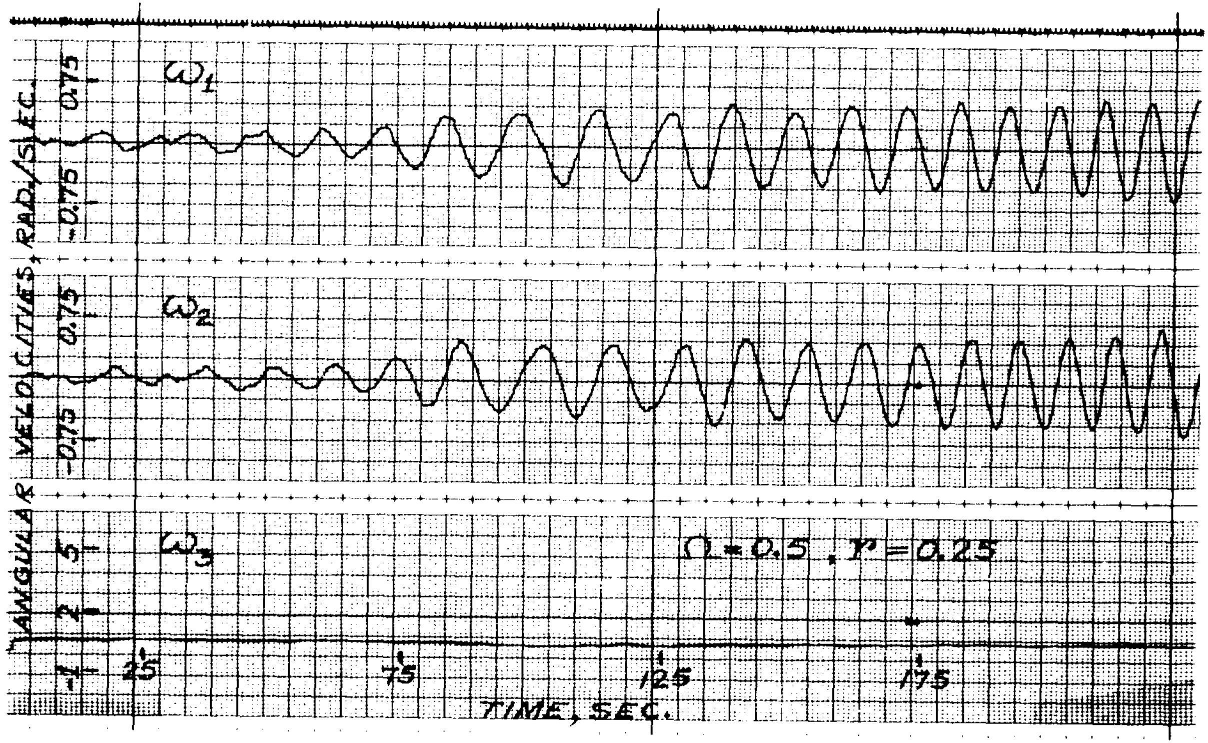
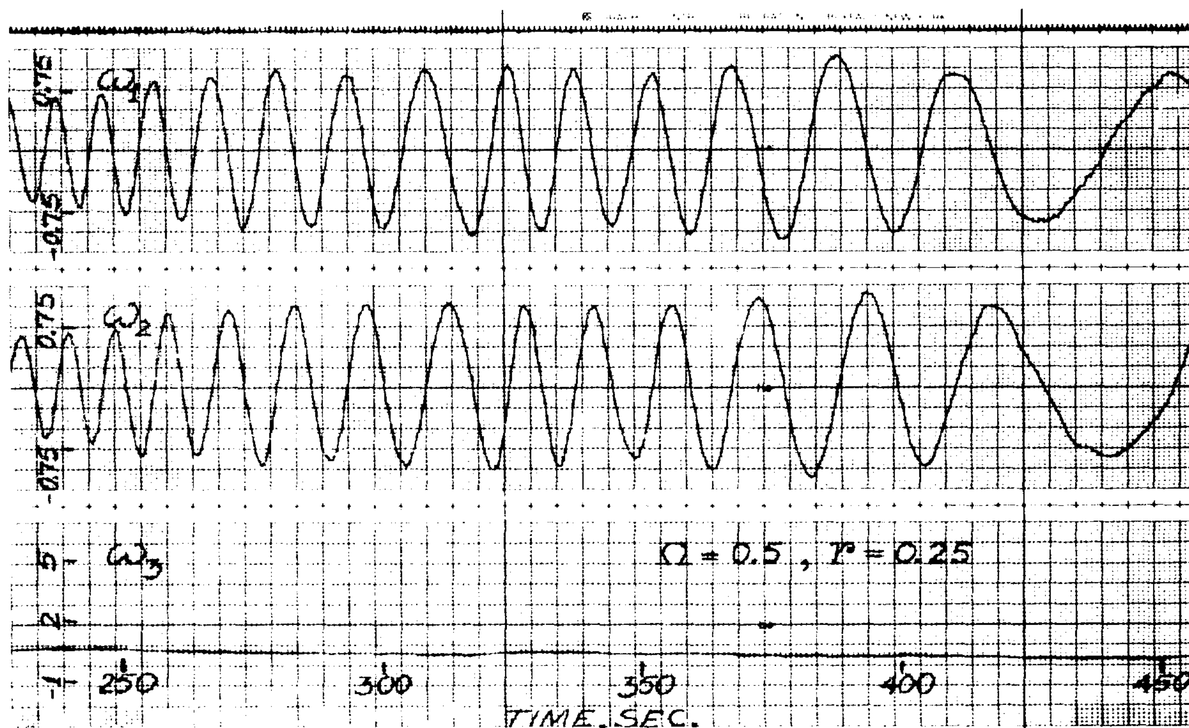

ANGULAR VELOCITIES, RAD./SEC.
ω_1
0.75
-0.75
ω_2
0.75
-0.75
ω_3
$\Omega = 0.5$, $\gamma = 0.25$
25
75
125
175
TIME, SEC.

Figure 10.



ω_1
0.75
-0.75
ω_2
0.75
-0.75
ω_3
5
2
-1
$\Omega = 0.5$, $\Gamma = 0.25$
250
300
350
400
450
TIME, SEC.

Figure 11.

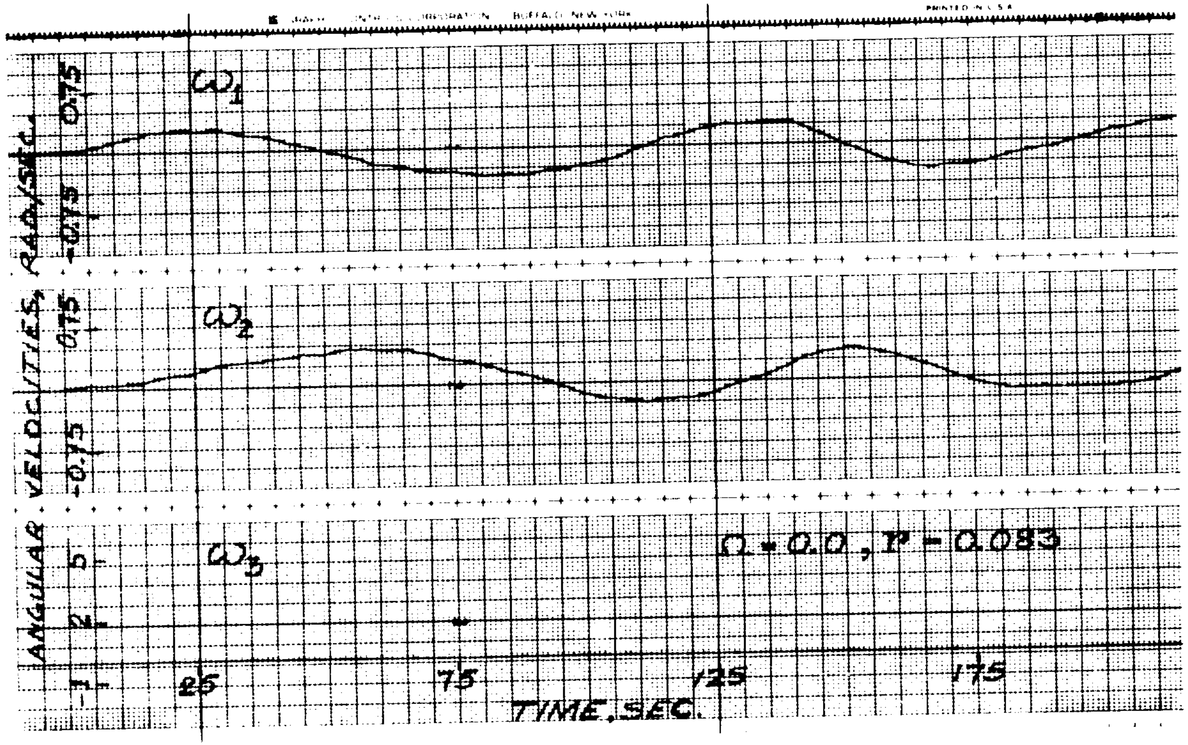

ANGULAR VELOCITIES, RAD/SEC.
0.75
-0.75
ω_1
ω_2
ω_3
5
2
1
$\Omega = 0.0$, $\Gamma = 0.083$
25
75
125
175
TIME, SEC.

Figure 12.

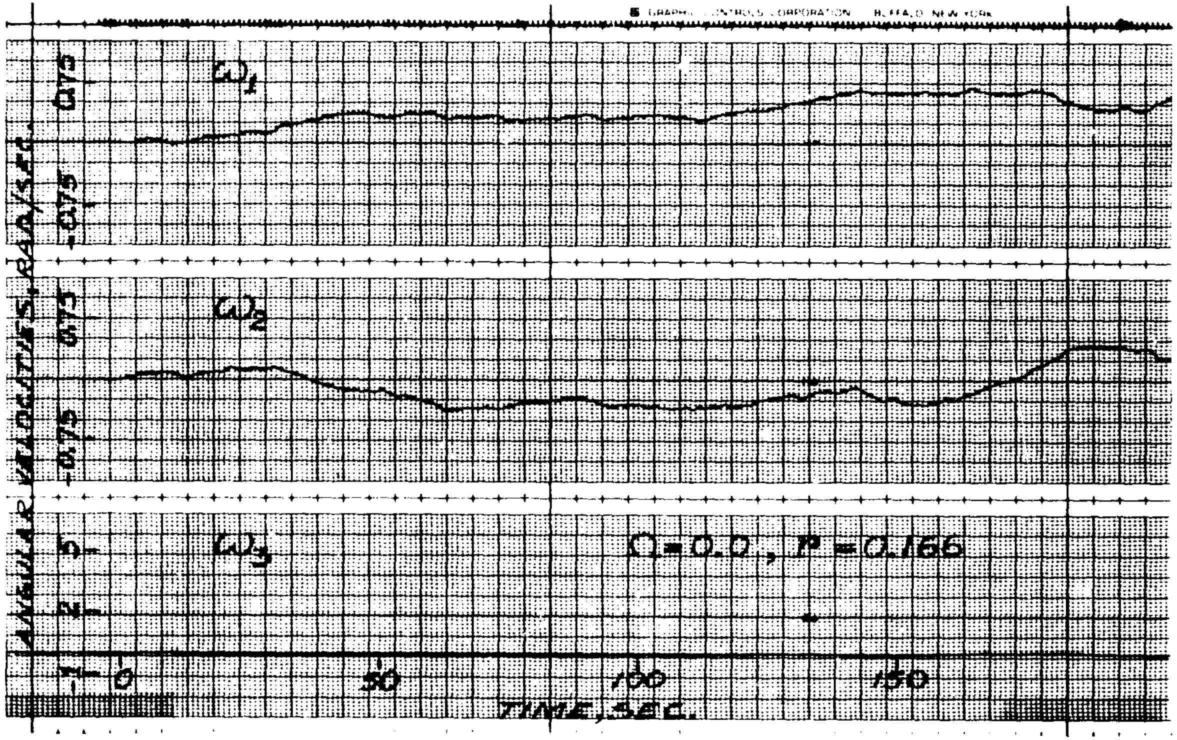
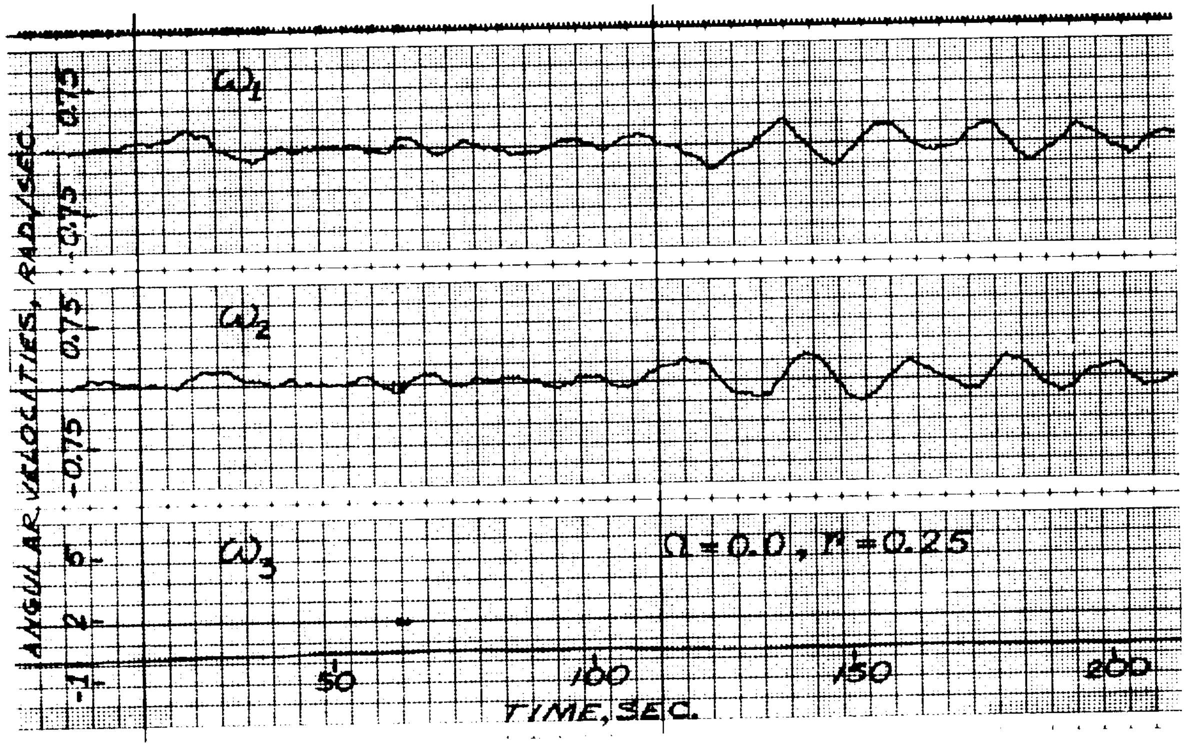

ANGULAR VELOCITIES, RAD/SEC.
0.75
-0.75
ω_1
ω_2
ω_3
Ω=0.0, P=0.166
0
50
100
150
TIME, SEC.

Figure 13.



ANGULAR VELOCITIES, RAD/SEC
0.75
-0.75
ω_1
ω_2
ω_3
6
2
-1
$\Omega = 0.0$, $r = 0.25$
50
100
150
200
TIME, SEC.

Figure 14.

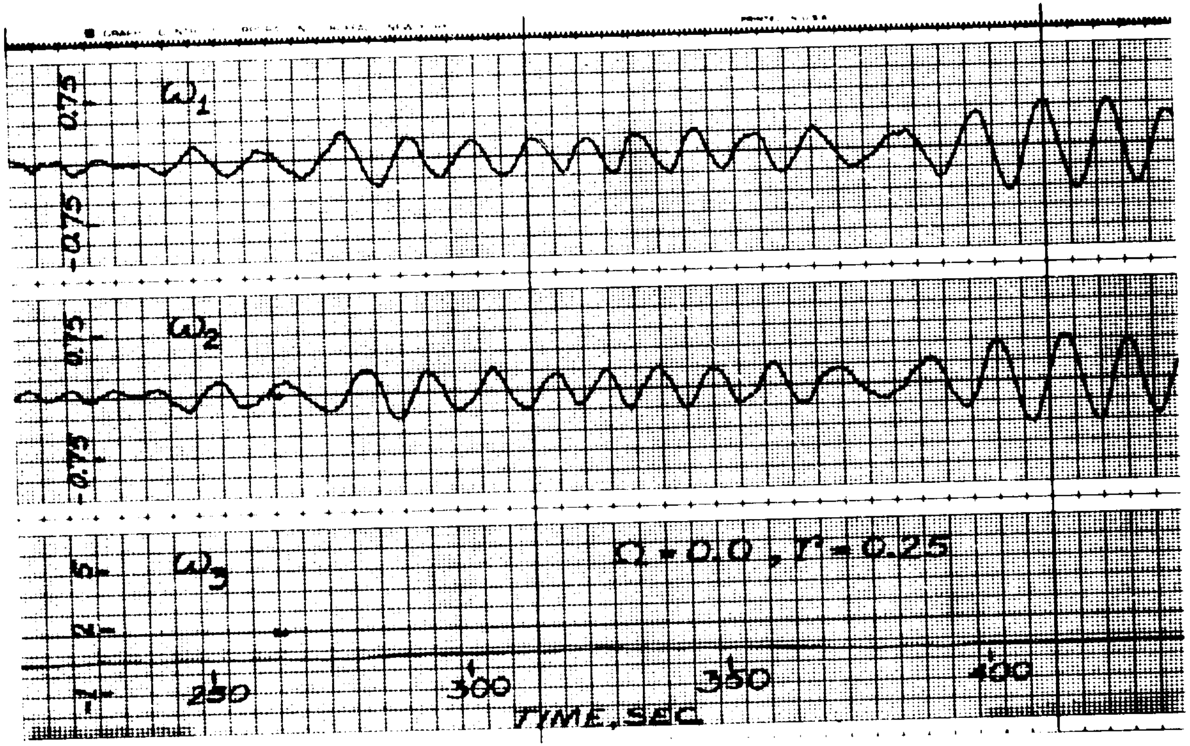

0.75
-0.75
ω_1
0.75
-0.75
ω_2
ω_3
$\Omega = 0.0$, $\Gamma = 0.25$
250
300
350
400
TIME, SEC

Figure 15.

STABILITY AND CONTROL OF FLEXIBLE SATELLITES

PART I - STABILITY

T. C. Huang and Aniruddha Das

ABSTRACT

This investigation has two distinct parts. In this first part the environmental and control torques experienced by a satellite are assumed to be random so as to account for the inherent errors in the control systems and the lack of exact models of the environmental torques. It has been shown that under this assumption the required stability criteria of a satellite is quite different from that obtained by a deterministic approach. It has also been shown that a flexible three-axes stabilized satellite can be made almost certainly asymptotically stable, while the same is not true for a flexible spinning satellite.

NOMENCLATURE

A^* = Composite body of a flexible satellite.

$[A_i]$, i = 1-5 = Matrices associated with the equations of motion of the flexible elements; Eqs. (3), (49), (53) - (57).

$[A_i']$, i = 1-5 = Matrices similar to $[A_i]$; Eq. (44).

a = Radius of the cylindrical rigid core of the assumed satellite configuration; Fig. 2.

a^* = Normalizing factor of the joint probability density; Eq. (17).

B^* = Additional composite body for a flexible dual-spin satellite.

$[B_1]$, $[B_2]$ = Matrices associated with combined equations of motion of the satellite; Eqs. (5) - (7).

b_1, b_3 = Elements of $\underline{r}_{d1}$, $\underline{r}_{d3}$; Eq. (38).

$[C]$ = Stochastic system matrix; Eqs. (21), (27).

C_i, i = 1-10 = Coefficients of the characteristic Eq. (71); Eqs. (74) - (77), (80), (85), (86).

e_i, i = 1-4 = Elements of $\underline{r}_{di}$, i = 1-4; Eq. (38).

$\underline{f}(t)$ = Deterministic forcing function; Eqs. (5), (10).

f_2, f_4 = Elements of $\underline{r}_{d2}$, $\underline{r}_{d4}$; Eq. (38).

$\underline{f}^*(t)$ = Deterministic environmental torque vector on the satellite; Eqs. (4), (45), (50).

$[G]$	= Stochastic control matrix, Eqs. (21), (26).
g_{ij}	= Elements of coefficient matrix defined by Eq. (63).
$\underline{h}[\underline{x}(t)]$	= Deterministic observed function of $\underline{x}(t)$; Eq. (13).
$[I]$	= Identity matrix.
$[\hat{I}]$	= Moment of inertia matrix of the nominal configuration of the satellite.
I_x, I_y, I_z	= Diagonal elements of $[\hat{I}]$; Eq. (52).
J	= The joint probability density of $(\underline{z}-\overline{\underline{z}})$, $(\underline{u}-\overline{\underline{u}})$, $(\underline{f}-\overline{\underline{f}})$ and $[x(0) - \overline{\underline{x}}(0)]$, Eq. (17).
J^*	= Functional defined by Eq. (17a).
J^{**}	= Functional defined by Eq. (18).
ℓ_i, $i = 1\text{-}4$	= Lengths of flexible beams of the satellite.
$[N_i]$, $i = 1\text{-}4$	= Submatrices of $[B_1]^{-1}$; Eqs. (28), (29).
$[0]$	= Null matrix.
$[P_0]$	= Covariance matrix of $[\underline{x}(0) - \overline{\underline{x}}(0)]$; Eq. (12).
$[P_i]$, $i = 1\text{-}5$	= Matrices associated with the angular momentum equations of the flexible satellite; Eqs. (4), (50).
$[P_i']$, $i = 1\text{-}5$	= Matrices similar to $[P_i]$; Eq. (45).
p_i	= Eigenvalues of $[-B_1^{-1}B_2]$.
p_i^*, $i = 1\text{-}4$	= Exponents of the assumed beam displacement function; Eq. (42).
$[Q]$	= Covariance matrix of $[\underline{u}(t) - \overline{\underline{u}}(t)]$; Eq. (15).
Q_{ij}, $i,j=1\text{-}6$	= Elements of the characteristic matrix of Eqs. (49), (50); Eqs. (70), (71).
$\underline{q}$	= Generalized position vector of the flexible elements of the satellite; Eqs. (3), (4), (49), (50).
$\underline{q}'$	= Vector, similar to $\underline{q}$; Eqs. (44), (45).
$\underline{q}_{bi}$, $i = 1\text{-}4$	= Time dependent part of $\underline{y}_{bi}$; Eq. (42).
$[R]$	= Covariance matrix of $[\underline{z}(t) - \overline{\underline{z}}(t)]$; Eq. (14).
$\underline{r}_c$	= Displacement vector of the center of mass of the flexible satellite from its nominal position; Eq. (63).

$\underline{r}_{di}$, i = 1-4 = Nominal position vectors of the spring-mass-damper systems; Eq. (38).

$\underline{r}_{ri}$, i = 1-4 = Nominal position vectors of the beam-end masses; Eq. (37).

$[S^K]$, K = 1-3 = Coefficient matrix; Eq. (67).

S = Covariance matrix of $[\underline{f}(t) - \overline{\underline{f}}(t)]$; Eq. (16).

$\underline{s}$ = Generalized velocity of the flexible elements; Eq. (8), (11).

T = Terminal point of controlling time interval

$\overline{T}$ = Terminal point of the time interval in which the maximum likelihood estimates are required.

$\hat{T}$ = Total kinetic energy of the flexible satellite.

t = Time.

$\underline{u}$ = Augmented control torque vector; Eqs. (5), (9).

$\underline{u}^*$ = Control torque vector; Eqs. (4), (45), (50).

$\underline{X}$ = Stochastic system state variable; Eqs. (21), (24).

$\underline{x}$ = Deterministic system state variable; Eqs. (5), (8).

$\underline{y}_{bi}$, i = 1-4 = Displacement vector of beams.

y_{di}, i = 1-4 = Components of $\underline{y}^*_{di}$; Eq. (41).

$\underline{y}^*_{di}$, i = 1-4 = Displacement vector of spring-mass-damper systems.

$\underline{y}_{ri}$, i = 1-4 = Displacement vector of beam-end masses.

$\underline{Z}$ = Stochastic forcing function; Eqs. (21), (25).

$\underline{z}$ = Observed values of the state variables; Eqs. (13), (19), (23).

α = Characteristic values of Eqs. (49), (50); Eq. (70).

$[\alpha^k]$, k = 0-3 = Coefficients of structural equations; Eqs. (64) - (68).

$\delta(t)$ = Dirac's delta function.

$\underline{\theta}$ = Relative angular displacement vector of A* with respect to B*.

$\underline{\lambda}$ = Lagrangian multiplier and state variable; Eqs. (18), (22).

$\underline{\mu}$ = Lagrangian multiplier; Eq. (18).

μ^k_{bij} = Coefficients of structural equations; Eqs. (64) - (66).

τ_{bij}	= Beam displacement mode parameters; Eqs. (46), (47).
τ_{di}	= Spring-mass-damper displacement mode parameters; Eq. (48).
$\underline{\Omega}$	= Nominal angular velocity vector of the satellite.
$\underline{\omega}$	= Perturbed angular velocity vector; Eq. (2).
$\underline{\omega}^*$	= Angular velocity vector of the satellite.
$\underline{\omega}_A^*$	= Angular velocity vector of A*.
$\underline{\omega}_B^*$	= Angular velocity vector of B*.

OPERATORS

$(\dot{\ })$	= Time derivative; $\frac{d}{dt}$.
$[\ ]^T$	= Transpose.
$(\tilde{\ })$	= Vector cross product operator; Eq. (69).
$(\bar{\ })$	= Mean value.
$(\underline{\ })$	= Vector.
Det. [ ]	= Determinant of the matrix.
E[ ]	= Statistical expectation.
Tr. [ ]	= Trace of the matrix.

INTRODUCTION

The primary requirement of an artificial satellite is that it should be capable of precise orientation in space. This capability is determined mainly by the stability and controllability of the satellite when viewed as a dynamic system. A large number of investigations have been made in the area of flexible satellite dynamics. But several interesting questions on the stability and controllability of flexible satellites have not been examined in sufficient detail. The present study looks at two of these questions:

(a) What are the stability criteria of flexible satellites in the presence of errors in the controlling torques and largely unknown environmental torques?

(b) For a given control system, and for a given number of torquing jets, is it possible to increase the controllability of a flexible satellite by monitoring the deflections of the flexible elements?

In the first part of this study it will be shown that, in the presence of random errors in the external torques on a flexible satellite, the stability criteria are far more restrictive than those deduced from a deterministic approach. The second part of this study will present reasons for an affirmative answer to question (b).

As mentioned earlier, deterministic criteria for the stability of flexible satellites have been studied extensively [1-4]. It must be noted that, to account for errors in the external torques acting on the satellite, these torques and the dynamic state variables of the satellite model must be treated as stochastic variables. Several studies [5-7] on the state identification problem have been done. These studies generally assumed Gaussian distributions and used Kalman filtering techniques. Using methods similar to that given in Ref. [8], equations of motion and the stochastic angular velocity response of flexible satellites have been computed in Refs. [9,10]. But the problem of comparing the stability characteristics of various satellite configurations subjected to random excitations has not been investigated.

DETERMINISTIC EQUATIONS OF MOTION

Formal deterministic equations of motion of a flexible satellite can be established. The stochastic stability boundaries can be determined only when these equations are available.

Let $\underline{\omega}^*(t)$ be the angular velocity vector of a flexible satellite. For a single body satellite, $\underline{\omega}^*(t)$ is a (3x1) vector. For a dual-spin satellite with two main composite bodies (A^* and B^*), $\underline{\omega}^*(t)$ is usually taken as

$$\underline{\omega}^*(t) = [\underline{\omega}_A^*(t),\ \underline{\omega}_B^*(t),\ \underline{\dot{\theta}}(t)]^T \tag{1}$$

In the above equation, $\underline{\omega}_A^*$ and $\underline{\omega}_B^*$ are the (3x1) angular velocity vectors of the composite bodies A^* and B^*; while $\underline{\dot{\theta}}$ is the (3x1) relative angular velocity vector of the body A^* with respect to B^*. Let $\underline{\Omega}$ be the constant vector of the nominal values of $\underline{\omega}^*(t)$, such that the perturbing angular velocity vector $\underline{\omega}(t)$ is defined by

$$\underline{\omega}(t) = \underline{\omega}^*(t) - \underline{\Omega} \tag{2}$$

Let the motions of the flexible elements of the satellite be represented by the generalized (nx1) position vector $\underline{q}(t)$. With these definitions, the equations of motion of the flexible elements can be expressed in the following form:

$$\begin{aligned}[A_1]\underline{\ddot{q}}(t) + [A_2(\underline{\omega},\underline{\dot{\omega}},\underline{\Omega},t)]\underline{\dot{q}}(t) + [A_3(\underline{\omega},\underline{\dot{\omega}},\underline{\Omega},t)]\underline{q}(t) \\ = [A_4]\underline{\dot{\omega}}(t) + [A_5(\underline{\omega},\underline{\Omega},t)]\underline{\omega}(t)\end{aligned} \tag{3}$$

Similarly, the equations for the conservation of angular momentum of the composite bodies of the satellite can be shown to be of the form:

$$\begin{aligned}[P_1]\underline{\ddot{q}}(t) + [P_2(\underline{\omega},\underline{\dot{\omega}},\underline{\Omega},t)]\underline{\dot{q}}(t) + [P_3(\underline{\omega},\underline{\dot{\omega}},\underline{\Omega},t)]\underline{q}(t) \\ = [P_4]\underline{\dot{\omega}}(t) + [P_5(\underline{\omega},\underline{\Omega},t)]\underline{\omega}(t) + \underline{u}^*(t) + \underline{f}^*(t)\end{aligned} \tag{4}$$

where $\underline{u}^*(t)$ and $\underline{f}^*(t)$ are the controlling and environmental torque vectors, respectively.

Detailed methods of developing Eqs. (3,4) are given in Refs. [1–4] and especially in Refs. [11,12]. Eqs. (3,4) provide the complete set of equations of motion of the flexible satellite. Equation (3) contains 'n' scalar equations, such that the matrices $[A_1]$, $[A_2]$ and $[A_3]$ are square. Equation (4) contains either three or nine equations depending on whether the satellite is of a single body or a dual-spin type.

Equations (3,4) can be combined in the form

$$[B_1]\dot{\underline{x}} + [B_2]\underline{x} = \underline{u}(t) + \underline{f}(t) \tag{5}$$

where, defining [I] to be the identity matrix,

$$[B_1] = \begin{bmatrix} P_4 & -P_1 & 0 \\ A_4 & -A_1 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{6}$$

$$[B_2] = \begin{bmatrix} P_5 & -P_2 & -P_3 \\ A_5 & -A_2 & -A_3 \\ 0 & -I & 0 \end{bmatrix} \tag{7}$$

$$\underline{x}(t) = [\underline{\omega}(t), \underline{s}(t), \underline{q}(t)]^T \tag{8}$$

$$\underline{u}(t) = [\underline{u}^*(t), \underline{0}, \underline{0}]^T \tag{9}$$

$$\underline{f}(t) = [\underline{f}^*(t), \underline{0}, \underline{0}]^T \tag{10}$$

and

$$\underline{s}(t) = \dot{\underline{q}}(t) \tag{11}$$

Equation (5) is the required differential equation describing the deterministic motions of a flexible satellite.

STOCHASTIC EQUATIONS OF MOTION

The stochastic equations of motion of the flexible satellite will now be obtained following the method shown in Refs. [8,9].

Let it be assumed that the initial values, $\underline{x}(0)$, have a Gaussian distribution with a known mean value, $\underline{\bar{x}}(0)$, and a known covariance matrix, $[P_0]$, given by

$$[P_0] = E\{[\underline{x}(0) - \underline{\bar{x}}(0)][\underline{x}(0) - \underline{\bar{x}}(0)]^T\} \tag{12}$$

Here the operator E denotes statistical expectation. Let $\underline{x}(t)$ be monitored on the Earth by measuring a variable $\underline{z}(t)$ where the mean value, $\underline{\bar{z}}(t)$, of $\underline{z}(t)$ is related to $\underline{x}(t)$ by

$$\underline{\bar{z}}(t) = \underline{h}[\underline{x}(t)] \tag{13}$$

Let it also be assumed that the variables $\underline{z}(t)$, $\underline{u}(t)$, and $\underline{f}(t)$ are Gaussian with known mean values and covariance matrices $R(t)$, $Q(t)$, and $S(t)$, respectively. Hence, assuming zero lag, we get

$$E\{[\underline{z}(t) - \underline{\bar{z}}(t)][\underline{z}(\tau) - \underline{\bar{z}}(\tau)]^T\} = R(t)\delta(t-\tau) \tag{14}$$

$$E\{[\underline{u}(t) - \underline{\bar{u}}(t)][\underline{u}(\tau) - \underline{\bar{u}}(\tau)]^T\} = Q(t)\delta(t-\tau) \tag{15}$$

$$E\{(\underline{f}(t) - \underline{\bar{f}}(t)][\underline{f}(\tau) - \underline{\bar{f}}(\tau)]^T\} = S(t)\delta(t-\tau) \tag{16}$$

where $\underline{\bar{u}}(t)$ and $\underline{\bar{f}}(t)$ are the mean values of $\underline{u}(t)$ and $\underline{f}(t)$, respectively.

Let the maximum-likelihood estimates of the response of the satellite be required in the time interval $[0,\bar{T}]$. In view of the definitions given above, the joint probability density, J, of $(\underline{z}-\underline{\bar{z}})$, $(\underline{u}-\underline{\bar{u}})$, $(\underline{f}-\underline{\bar{f}})$ and $[\underline{x}(0) - \underline{\bar{x}}(0)]$ is given by

$$J = a^*[\exp(-\frac{J^*}{2})] \tag{17}$$

where J^* is defined as

$$\begin{aligned} J^* = {} & [\underline{x}(0) - \underline{\bar{x}}(0)]^T[P_0]^{-1}[\underline{x}(0) - \underline{\bar{x}}(0)] \\ & + \int_0^{\bar{T}} [\underline{z}(t) - \underline{\bar{z}}(t)]^T[R(t)]^{-1}[\underline{z}(t) - \underline{\bar{z}}(t)] \\ & + [\underline{u}(t) - \underline{\bar{u}}(t)]^T[Q(t)]^{-1}[\underline{u}(t) - \underline{\bar{u}}(t)] \\ & + [\underline{f}(t) - \underline{\bar{f}}(t)]^T[S(t)]^{-1}[\underline{f}(t) - \underline{\bar{f}}(t)]\, dt \end{aligned} \tag{17a}$$

and 'a*' is the normalizing factor.

The maximum-likelihood estimates can be obtained by maximizing the probability density J. In other words, we minimize the functional J^*, subject to the constraints that Eqs. (5), (13) be satisfied. This is done by defining J^{**} by the relation

$$\begin{aligned} J^{**} = J^* + 2 \int_0^{\bar{T}} \{ & \underline{\mu}^T[\underline{\bar{z}}(t) - \underline{h}(\underline{x})] \\ & + \underline{\lambda}^T\ [\dot{\underline{x}} + B_1^{-1}\{B_2\underline{x}-\underline{u}-\underline{f}\}]\}dt \end{aligned} \tag{18}$$

and minimizing J^{**} by considering $\underline{x}(0)$, $\underline{\bar{z}}(t)$, $\underline{u}(t)$, $\underline{f}(t)$, $\underline{x}(t)$ and the Lagrangian vector multipliers $\underline{\mu}(t)$ and $\underline{\lambda}(t)$ as the independent variables.

It will now be assumed that

$$\underline{z}(t) = \underline{h}[\underline{x}(t)] = \underline{x}(t) \tag{19}$$

which means

$$\frac{\partial \underline{h}}{\partial \underline{x}} = [I] \tag{20}$$

With this assumption, the variational equations obtained by minimizing J** are expressed as

$$\dot{\underline{X}}(t) = [C]\underline{X}(t) + [G]\overline{\underline{u}}(t) + \underline{Z}(t) \quad , \tag{21}$$

$$\underline{\lambda}(T) = 0 \tag{22}$$

and

$$\underline{x}(0) = \overline{\underline{x}}(0) + [P_0]\underline{\lambda}(0) \tag{23}$$

where

$$\underline{X}(t) = [\underline{x}(t), \underline{\lambda}(t)]^T \tag{24}$$

$$\underline{Z}(t) = [B_1^{-1}\ \underline{f}(t), - R^{-1}\underline{z}(t)] \tag{25}$$

$$[G] = \begin{bmatrix} B_1^{-1} \\ \\ 0 \end{bmatrix} \tag{26}$$

$$[C] \quad \begin{bmatrix} -B_1^{-1}B_2 & B_1^{-1}[Q+S][B_1^{-1}]^T \\ \\ R^{-1} & [B_1^{-1}B_2]^T \end{bmatrix} \tag{27}$$

Equations (21-23) are the required stochastic differential equations of motion of the flexible satellite.

STABILITY CRITERIA

The stochastic Eq. (21) has twice as many scalar equations as the deterministic Eq. (5). The deterministic equations are stable if the eigenvalues of $[-B_1^{-1}B_2]$ have negative real parts. The stochastic equations are stable if all the eigenvalues of [C] have negative real parts. If there were no errors involved with $\underline{u}(t)$ and $\underline{f}(t)$, the matrices [Q] and [S] would be null matrices. Consequently, Eq. (21) would degenerate into Eq. (5).

The hypothesis of this study is that [Q] and [S] are not null matrices, but have positive elements which are very small compared to those of $[B_1]$ or $[B_2]$. Hence, half of the eigenvalues of [C] will be almost equal to the eigenvalues, p_i, of $[-B_1^{-1}B_2]$ and the other half will be almost equal to $-p_i$. That the eigenvalues of [C] lie symmetrically about the imaginary axis can be verified by noting that

$$Tr[C] = 0$$

and that the eigenvalues of $[B_1^{-1}B_2]^T$ are equal and opposite to those of $[-B_1^{-1}B_2]$.

In view of this, it is evident that Eq. (21) is always unstable. Even if the real parts of p_i are zero, the instability will be caused by the multiple roots. Thus, according to the classical meaning of the term, no stability criterion exists for the stochastic Eq. (21). The physical reason behind this is that the probable errors in the dependent variables accumulate with time. This accumulation causes the maximum-likelihood estimates to be asymptotically divergent, even if the deterministic Eq. (5) is stable. The growth phenomenon, for a satellite in which the vector $\underline{x}(t)$ is measured at discrete intervals of time, is illustrated in Figure 1. Let the mean values of $\underline{x}(t)$ be considered to be given by the solutions of Eq. (5). Let the variances of $\underline{x}(t)$ be computed from the differences of the values of $\underline{x}(t)$ computed from Eqs. (5) and (21). The error functions computed from these mean values and variances are shown at three instants of time in Figure 1. In Figure 1A, there is a data input and the computation cycle has been started. Hence the error distribution curve has a high peak. The variances here correspond only to the measurement errors of the variables $\underline{x}(t)$. In Figures 1B and 1C, it is seen that the height of the error function becomes shorter and shorter, although the mean position given by Eq. (5) approaches the origin. In Figure 1C, the error function is very flat just before the new data input. It becomes sharp again just after the new data input when a new computation cycle is started.

Since Eq. (21) is necessarily unstable, the stochastic stability cirteria for a flexible satellite must be formulated in a particular manner. The stochastic stability criteria of the response of a flexible satellite are those which make

(a) the deterministic model given by Eq. (5) stable, and

(b) the growth rate of the stochastic model given by Eq. (21) a minimum.

In the absence of further information about the covariance matrices Q, R and S, these two requirements are met if the real parts of p_i are equal to zero. Thus, a flexible satellite will be called stochastically stable if all the eigenvalues of $[-B_1^{-1}B_2]$ are purely imaginary. It is interesting to note at this point that a perfectly rigid satellite satisfies this requirement.

Specific stability criteria can be obtained for a satellite when the elements of $[B_1]$ and $[B_2]$ are known. For this, a particular satellite configuration has to be assumed. In the absence of such a specific configuration, several conditions sufficient to make the p_i purely imaginary can be established in terms of the matrices $[A_i]$ and $[P_i]$, $i = 1$-5, when the matrices are square.

SUFFICIENT CONDITIONS

The sufficient condition for the p_i to be purely imaginary, the matrix $[B_1^{-1}B_2]$ must be antisymmetric. Let $[A_i]$ and $[P_i]$, $i = 1-5$, be square matrices. Let $[B_1^{-1}]$ be given by

$$[B_1]^{-1} = \begin{bmatrix} N_1 & N_2 & 0 \\ N_3 & N_4 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{28}$$

Comparing Eqs. (6) and (28), the matrices N_i, $i = 1-4$, are given by

$$\begin{aligned} [N_1] &= [P_4 - P_1A_1^{-1}A_4]^{-1} \\ [N_2] &= [A_4 - A_1P_1^{-1}P_4]^{-1} \\ [N_3] &= [P_4A_4^{-1}A_1-P_1]^{-1} \\ [N_4] &= [A_4P_4^{-1}P_1-A_1]^{-1} \quad . \end{aligned} \tag{29}$$

Hence from Eqs. (7) and (28), $[B_1^{-1}B_2]$ is given by

$$[B_1^{-1}B_2] = \begin{bmatrix} [N_1P_5+N_2A_5] & -[N_1P_2+N_2A_2] & -[N_1P_3+N_2A_3] \\ [N_3P_5+N_4A_5] & -[N_3P_2+N_4A_2] & -[N_3P_3+N_4A_3] \\ [0] & [-I] & [0] \end{bmatrix} \tag{30}$$

To have $[B_1^{-1}B_2]$ antisymmetric, the required conditions become

$$\begin{aligned} N_1P_5 + N_2A_5 &= 0 \\ N_3P_2 + N_4A_2 &= 0 \\ N_1P_3 + N_2A_3 &= 0 \\ N_3P_3 + N_4A_3 &= -I \\ N_1P_2 + N_2A_2 &= N_3P_5 + N_4A_5 \quad . \end{aligned} \tag{31}$$

Eliminating N_i, i = 1-4 from Eqs. (29) and (31), the required sufficient conditions are finally obtained as

$$[P_3] = [P_1] \tag{32}$$

$$[A_3] = [A_1] \tag{33}$$

$$[P_2] = -[P_4A_1^{-1}A_5] \tag{34}$$

$$[A_2] = -[A_4A_1^{-1}A_5] \tag{35}$$

$$[P_5] = [P_1A_1^{-1}A_5] \tag{36}$$

The stochastic stability criteria given by Eqs. (32-36) are much too restrictive and it will be almost impossible to obtain a practical design of a satellite satisfying these constraints. For example, Eq. (33) requires that the natural frequencies of the flexible elements of the satellite should be equal to unity. This is not a feasible constraint.

In spite of these drawbacks, Eqs. (32-36) do provide several guidelines for satellite design. It can be easily verified that Eqs. (34-36) are satisfied identically by a three-axes stabilized satellite in which all subbodies have undamped, purely elastic mountings. A spinning or a dual-spin satellite, even if it is free of damping, generally does not satisfy Eqs. (34-36). Equation (32) is satisfied by all types of satellites in which there is an axis of symmetry, and in which the flexible elements are so constrained that the center of mass moves only along the axis of symmetry. Hence it can be claimed that, among satellite designs with comparable mass, stiffness, damping and covariance matrices, a symmetric, three-axes stabilized satellite is likely to have the lowest error growth rate.

A SPECIFIC CONFIGURATION

The constraints given by Eqs. (32-36) are too restrictive because, in their derivation, no attention has been paid to the zero elements of the matrices involved. To utilize the location of the zero elements in the matrices $[B_1]$ and $[B_2]$, a particular satellite configuration (shown in Figure 2) will now be considered. The satellite consists of a rigid cylindrical body with four beams, four beam-tip masses, and four spring-mass-damper systems, placed symmetrically as required by Eq. (32). The beams are perpendicular to the axis of symmetry and are assumed to be axially rigid. The spring-mass-damper systems are assumed to be constrained to move only parallel to the axis of symmetry. These assumptions lead to a large number of zeros in the matrices $[B_1]$ and $[B_2]$, making the algebraic manipulations considerably simpler.

The major drawback of any stability analysis with a particular satellite configuration is that conclusions drawn from it cannot be extended to other configurations. The method of modelling and analysis of the satellite configuration (shown in Figure 2) that has been used in this study partially overcomes this disadvantage. In this method, the location of zeros in $[B_1]$ and $[B_2]$ remain unchanged when the numbers of beams, tip-masses, or spring-mass-damper systems are changed.

THE DYNAMIC MODEL

Let 'a' be the radius of the main rigid body and ℓ_i, i = 1-4, be the lengths of the beams. Let $\underline{r}_{ri}$ and $\underline{r}_{di}$, i = 1-4, be the nominal position coordinates of the beam-tip masses and the spring-mass-damper systems, respectively. According to the choice of coordinate axes shown in Figure 2, we have

$$
\begin{aligned}
\underline{r}_{r1} &= [(a+\ell_1),\ 0,\ 0]^T \\
\underline{r}_{r2} &= [0,\ -(a+\ell_2),\ 0]^T \\
\underline{r}_{r3} &= [-(a+\ell_3),\ 0,\ 0]^T \\
\underline{r}_{r4} &= [0,\ (a+\ell_4),\ 0]^T
\end{aligned}
\qquad (37)
$$

Let it be defined that

$$
\begin{aligned}
\underline{r}_{d1} &= [b_1,\ 0,\ e_1]^T \\
\underline{r}_{d2} &= [0,\ -f_2,\ e_2]^T \\
\underline{r}_{d3} &= [-b_3,\ 0,\ e_3]^T \\
\underline{r}_{d4} &= [0,\ f_4,\ e_4]^T
\end{aligned}
\qquad (38)
$$

Let x be the distance along the axes of the beams measured from the fixed ends. Let $\underline{y}_{ri}(t)$, $\underline{y}_{bi}(x,t)$ and $\underline{y}^{*}_{di}(t)$, i = 1-4 be the deflections of the beam-tip masses, the beams, and the spring-mass-damper systems, respectively. According to the previously assumed constraints, let it be defined that

$$
\begin{aligned}
\underline{y}_{r1}(t) &= [0,\ y_{r1,2}(t),\ y_{r1,3}(t)]^T \\
\underline{y}_{r2}(t) &= [y_{r2,1}(t),\ 0,\ y_{r2,3}(t)]^T \\
\underline{y}_{r3}(t) &= [0,\ y_{r3,2}(t),\ y_{r3,3}(t)]^T \\
\underline{y}_{r4}(t) &= [y_{r4,1}(t),\ 0,\ y_{r4,3}(t)]^T
\end{aligned}
\qquad (39)
$$

$$\underline{y}_{b1}(x,t) = [0,\ y_{b1,2}(x,t),\ y_{b1,3}(x,t)]^T$$
$$\underline{y}_{b2}(x,t) = [y_{b2,1}(x,t),\ 0,\ y_{b2,3}(x,t)]^T$$
$$\underline{y}_{b3}(x,t) = [0,\ y_{b3,2}(x,t),\ y_{b3,3}(x,t)]^T \tag{40}$$
$$\underline{y}_{b4}(x,t) = [y_{b4,1}(x,t),\ 0,\ y_{b4,3}(x,t)]^T$$

and

$$\underline{y}^*_{di}(t) = [0,\ 0,\ y_{di}(t)]^T \tag{41}$$

Equations of motion in the coordinates $\underline{\omega}$, $y_{ri,j}$, $y_{bi,j}$ and y_{di} for $i = 1\text{-}4$, $j = 1,3$ are obtained using the method shown in Ref. [1]. The space dependence of these equations is eliminated by assuming

$$y_{bi,j}(x,t) = [q_{bi,j}(t)][\exp(p_i^* x) - p_i^* x - 1] \tag{42}$$

and applying the Galerkin's method [1,11]. The space-dependent shape functions in Eq. (42) are assumed to be known and correspond to those of a cantilever beam with a tip-mass.

At this point, the boundary conditions

$$y_{ri,j}(t) = [\exp(p_i^* \ell_i) - p_i \ell_i - 1]q_{bi,j} \tag{43}$$

are applied, and the equations of motion reduce to the form

$$[A_1']\ddot{\underline{q}}'(t) + [A_2'(\underline{\omega},\dot{\underline{\omega}},\Omega,t)]\dot{\underline{q}}'(t) + [A_3'(\underline{\omega},\dot{\underline{\omega}},\Omega,t)]\underline{q}'(t)$$
$$= [A_4']\dot{\underline{\omega}}(t) + [A_5'(\underline{\omega},\Omega)]\underline{\omega}(t) \tag{44}$$

and

$$[P_1']\ddot{\underline{q}}'(t) + [P_2'(\underline{\omega},\dot{\underline{\omega}},\Omega,t)]\dot{\underline{q}}'(t) + [P_3'(\underline{\omega},\dot{\underline{\omega}},\Omega,t)]\underline{q}'(t)$$
$$= [P_4']\dot{\underline{\omega}}(t) + [P_5'(\underline{\omega},\Omega)]\underline{\omega}(t) + \underline{u}^*(t) + \underline{f}^*(t) \tag{45}$$

where $\underline{q}'(t)$ consists of the non-zero elements of $\underline{q}_{bi}$ and y_{di}, $i = 1\text{-}4$. The set of Eqs. (44) and (45) is of the order of 27. It is still quite difficult to extract any meaningful analytic stability criterion out of this set.

It is now assumed that there exists certain unknown constants τ_{bij} and τ_{di}, $i = 1\text{-}4$, $j = 1,3$, such that

$$\tau_{b12}q_{b1,2} = \tau_{b21}q_{b2,1} = \tau_{b32}q_{b3,2} = \tau_{b41}q_{b4,1} \tag{46}$$

$$\tau_{b13}q_{b1,3} = \tau_{b23}q_{b2,3} = \tau_{b33}q_{b3,3} = \tau_{b43}q_{b4,3} \tag{47}$$

and

$$\tau_{d1}y_{d1} = \tau_{d2}y_{d2} = \tau_{d3}y_{d3} = \tau_{d4}y_{d4} \tag{48}$$

The values of τ_{bij} and τ_{di} can be obtained from the eigenvectors of Eqs. (44), (45). But it is not our intention at this point to look for eigenvalues and eigenvectors of Eqs. (44), (45). Substituting Eqs. (46), (47), and (48) into Eqs. (44) and (45), the equations of motion of the satellites are reduced to the form

$$[A_1]\ddot{\underline{q}}(t) + [A_2]\dot{\underline{q}}(t) + [A_3]\underline{q}(t) = [A_4]\dot{\underline{\omega}} + [A_5]\underline{\omega} \tag{49}$$

and

$$[P_1]\ddot{\underline{q}}(t) + [P_2]\dot{\underline{q}}(t) + [P_3]\underline{q}(t) = [P_4]\dot{\underline{\omega}} + [P_5]\underline{\omega} + \underline{u}^*(t) + \underline{f}^*(t) \tag{50}$$

where

$$\underline{q}(t) = [y_{d1}, q_{b1,2}, q_{b1,3}]^T \tag{51}$$

It should be noted that $\underline{q}(t)$ given by Eq. (51) is a (3x1) vector and all matrices $[A_i]$ and $[P_i]$, $i = 1\text{-}5$, are (3x3) matrices. The Eqs. (49), (50) now form only a ninth order set of ordinary differential equations. This great reduction was made possible by the assumptions of Eqs. (46), (47), (48). It should also be noted that, irrespective of the number of beams or spring-mass-damper systems introduced at the initial stages of the dynamic modelling, Eqs. (49) and (50) can always be made a ninth order set by suitably augmenting the equations in Eqs. (46), (47), and (48).

Let it be assumed that the moment of inertia matrix, $[\hat{I}]$, of the satellite is given by

$$[\hat{I}] = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{52}$$

The linearized form of the matrices $[A_i]$ and $[P_i]$, $i = 1\text{-}5$, can then be shown to be as follows:

$$[A_1] = \begin{bmatrix} 0 & (\mu^1_{b12} - g_{11}\mu^3_{b12} - g_{21}\mu^4_{b12}) & 0 \\ -g_{33}\mu^3_{b13} & 0 & (\mu^1_{b13} - g_{32}\mu^3_{b13}) \\ (\mu^1_{d1} - g_{33}\mu^4_{d1} & 0 & -g_{32}\mu^4_{d1} \end{bmatrix} \tag{53}$$

$$[A_2] = \begin{bmatrix} 0 & -(g_{11}\mu_{b12}^{5} + g_{21}\mu_{b12}^{6}) & 0 \\ 0 & 0 & 0 \\ \mu_{d1}^{2} & 0 & 0 \end{bmatrix} \tag{54}$$

$$[A_3] = \begin{bmatrix} 0 & \mu_{b12}^{2} - g_{11}\mu_{b12}^{7} - g_{21}\mu_{b12}^{8}) & 0 \\ 0 & 0 & \mu_{b13}^{2} \\ \mu_{d1}^{3} & 0 & 0 \end{bmatrix} \tag{55}$$

$$[A_4] = \begin{bmatrix} 0 & 0 & \mu_{b12}^{9} \\ \mu_{b13}^{4} & \mu_{b13}^{5} & 0 \\ \mu_{d1}^{5} & \mu_{d1}^{6} & 0 \end{bmatrix} \tag{56}$$

$$[A_5] = \begin{bmatrix} 0 & 0 & 0 \\ \mu_{b13}^{6}\Omega_3 & \mu_{b13}^{7}\Omega_3 & 0 \\ \mu_{d1}^{7}\Omega_3 & \mu_{d1}^{8}\Omega_3 & 0 \end{bmatrix} \tag{57}$$

$$[P_1] = \begin{bmatrix} (\alpha_{13}^{0}g_{33} - S_{11}^{4}) & \alpha_{12}^{0}g_{21} & (\alpha_{13}^{0}g_{32} - S_{13}^{4}) \\ (\alpha_{23}^{0}g_{33} - S_{21}^{4}) & \alpha_{12}^{0}g_{11} & (\alpha_{23}^{0}g_{32} - S_{23}^{4}) \\ 0 & (\alpha_{13}^{0}g_{11} + \alpha_{23}^{0}g_{21} - S_{32}^{4}) & 0 \end{bmatrix} \tag{58}$$

$$[P_2] = \begin{bmatrix} (S_{21}^{4} - \alpha_{23}^{0}g_{33}) & -\alpha_{12}^{0}g_{11} & (S_{23}^{4} - \alpha_{23}^{0}g_{32}) \\ (\alpha_{13}^{0}g_{33} - S_{11}^{4} & \alpha_{12}^{0}g_{21} & (\alpha_{13}^{0}g_{32} - S_{13}^{4}) \\ 0 & 0 & 0 \end{bmatrix} \Omega_3 \tag{59}$$

$$[P_3] = \begin{bmatrix} (S_{23}^1+\alpha_{23}^3 g_{33}) & \alpha_{23}^2 g_{21} & (S_{32}^3+\alpha_{23}^3 g_{32}) \\ -(S_{13}^1+\alpha_{13}^3 g_{33}) & -\alpha_{13}^1 g_{11} & (S_{31}^3-\alpha_{13}^3 g_{32}) \\ 0 & 0 & 0 \end{bmatrix} \Omega_3^2 \tag{60}$$

$$[P_4] = [\hat{I}] \tag{61}$$

$$[P_5] = \tilde{\Omega}[\hat{I}] - [\widetilde{\hat{I}\,\underline{\Omega}}]. \tag{62}$$

The undefined constants introduced in Eqs. (53) - (62) are defined by the following relations:

$$\underline{r}_c = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & 0 & 0 \\ 0 & g_{32} & g_{33} \end{bmatrix} \begin{Bmatrix} q_{b1,2} \\ q_{b1,3} \\ y_{d1} \end{Bmatrix} \tag{63}$$

where $\underline{r}_c$ is the displacement of the center of mass of the satellite from its nominal position, and

$$\mu_{b12}^1 \ddot{q}_{b1,2} + \mu_{b12}^2 q_{b1,2} = \mu_{b12}^3 \ddot{r}_{c,1} + \mu_{b12}^4 \ddot{r}_{c,2} + \mu_{b12}^5 \dot{r}_{c,1}$$
$$+ \mu_{b12}^6 \dot{r}_{c,2} + \mu_{b12}^7 r_{c,1} + \mu_{b12}^8 r_{c,2} + \mu_{b12}^9 \dot{\omega}_3 \tag{64}$$

$$\mu_{b13}^1 \ddot{q}_{b1,2} + \mu_{b13}^2 q_{b1,3} = \mu_{b13}^3 \ddot{r}_{c,3} + \mu_{b13}^4 \dot{\omega}_1 + \mu_{b13}^5 \dot{\omega}_2$$
$$+ \mu_{b13}^6 \omega_1 \omega_3 + \mu_{b13}^7 \omega_2 \omega_3 \tag{65}$$

$$\mu_{d1}^1 \ddot{y}_{d1} + \mu_{d1}^2 \dot{y}_{d1} + \mu_{d1}^3 y_{d1} = \mu_{d1}^4 \ddot{r}_{c,3} + \mu_{d1}^5 \dot{\omega}_1 + \mu_{d1}^6 \dot{\omega}_2$$
$$+ \mu_{d1}^7 \omega_1 \omega_3 + \mu_{d1}^8 \omega_2 \omega_3 \tag{66}$$

$$\frac{\partial \hat{T}}{\partial \underline{\omega}} = [\hat{I}]\underline{\omega} + r_{c,1}[\alpha_{ij}^1]\underline{\omega} + r_{c,2}[\alpha_{ij}^2]\underline{\omega} + r_{c,3}[\alpha_{ij}^3]\underline{\omega} + y_{d1}[S_{ij}^1]\underline{\omega}$$
$$+ q_{b1,2}[S_{ij}^2]\underline{\omega} + q_{b1,3}[S_{ij}^3]\underline{\omega} + [S_{ij}^4]\dot{\underline{q}} - [\alpha_{ij}^0]\dot{r}_c \tag{67}$$

where $\hat{T}$ is the kinetic energy functional [1] of the satellite. The operator

$(\tilde{\ })$ on any (3x1) vector $\underline{v}$ is defined by

$$\tilde{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{68}$$

such that the cross-product between any two arbitrary vectors $\underline{u}$ and $\underline{v}$ is given by

$$\underline{u} \times \underline{v} = \tilde{u}\,\underline{v} = -\tilde{v}\,\underline{u} \tag{69}$$

Analytic search for the eigenvalues of Eqs. (49) and (50) is now quite easy, because these form only a ninth order set. As in the elements of the matrices $[A_i]$ and $[P_i]$, $i = 1$-4, these eigenvalues are functions of the unknown constants τ_{bij} and τ_{di}. The method of analysis to be adopted now is to obtain the stability criteria in terms of τ_{bij} and τ_{di}. Then we must obtain the union of all criteria such that the resulting criteria become independent of τ_{bij} and τ_{di}.

EIGENVALUE EQUATIONS

The characteristic equation in α for Eqs. (49) and (50) is given by

$$\text{Det.} \begin{bmatrix} [P_5+\alpha P_4] & -[P_3+\alpha P_2+\alpha^2 P_1] \\ [A_5+\alpha A_4] & -[A_3+\alpha A_2+\alpha^2 A_1] \end{bmatrix} = 0 \tag{70}$$

With the help of Eqs. (53) - (62), it can be seen that Eq. (70) is of the form

$$\text{Det.} \begin{bmatrix} Q_{11} & Q_{12} & 0 & Q_{14} & Q_{15} & Q_{16} \\ Q_{21} & Q_{22} & 0 & Q_{24} & Q_{25} & Q_{26} \\ 0 & 0 & Q_{33} & 0 & Q_{35} & 0 \\ 0 & 0 & Q_{43} & 0 & Q_{45} & 0 \\ Q_{51} & Q_{52} & 0 & Q_{54} & 0 & Q_{56} \\ Q_{61} & Q_{62} & 0 & Q_{64} & 0 & Q_{66} \end{bmatrix} = 0 \tag{71}$$

It can be verified that the locations of the zeros of the matrix in Eq. (71) remain the same even if the number of beams or spring-mass-damper systems are increased.

Equation (71) can be factorized into

$$\text{Det.}\begin{bmatrix} Q_{11} & Q_{12} & Q_{14} & Q_{16} \\ Q_{21} & Q_{22} & Q_{24} & Q_{26} \\ Q_{51} & Q_{52} & Q_{54} & Q_{56} \\ Q_{61} & Q_{62} & Q_{64} & Q_{66} \end{bmatrix} [Q_{43}Q_{35}-Q_{33}Q_{45}) = 0$$

Thus, the characteristic equations become

$$(Q_{43}Q_{35}-Q_{33}Q_{45}) = 0 \tag{72}$$

and

$$\text{Det.}\begin{bmatrix} Q_{11} & Q_{12} & Q_{14} & Q_{16} \\ Q_{21} & Q_{22} & Q_{24} & Q_{26} \\ Q_{51} & Q_{52} & Q_{54} & Q_{56} \\ Q_{61} & Q_{62} & Q_{64} & Q_{66} \end{bmatrix} = 0 \tag{73}$$

Equation (72) yields three roots of α and the other six roots are obtained from Eq. (73). One of the roots of α from Eq. (72) is identically equal to zero. The other two roots of Eq. (72) are given by the equation

$$C_1\alpha^2 + C_2\alpha + C_3 = 0 \tag{74}$$

where

$$C_1 = [I_z(\mu^1_{b12}-g_{11}\mu^3_{b12}-g_{21}\mu^4_{b12}) + \mu^9_{b12}(S^4_{32}-\alpha^0_{13}g_{11}-\alpha^0_{23}g_{21})] \tag{75}$$

$$C_2 = -\, I_z(g_{11}\mu^5_{b12} + g_{21}\mu^6_{b12}) \tag{76}$$

$$C_3 = I_z(\mu^2_{b12}-g_{11}\mu^7_{b12}-g_{21}\mu^8_{b12}) \;. \tag{77}$$

Hence the requirement of purely imaginary roots leads to the conditions

$$C_2 = 0 \;;\; C_3/C_1 \geq 0 \quad \text{if } C_1 \neq 0 \;. \tag{78}$$

Expanding Eq. (73), the resulting equation in α is obtained as

$$C_4\alpha^6 + C_5\alpha^5 + C_6\alpha^4 + C_7\alpha^3 + C_8\alpha^2 + C_9\alpha + C_{10} = 0 \tag{79}$$

To simplify the expressions of C_i, i = 4-10, let it be assumed that

$$\tau_{b12} = -\tau_{b32}\,;\ \tau_{b21} = -\tau_{b41}\,;\ \tau_{b13} = -\tau_{b33}\,;\ \tau_{b23} = -\tau_{b43}\,;$$

$$\tau_{d1} = -\tau_{d3}\,;\ \tau_{d2} = -\tau_{d4} \qquad (79a)$$

This assumed mode corresponds to that which, in terms of pointing accuracy, we are most interested. This mode leads to pure rotational motions of the rigid core about its center of mass. With this assumption, the coefficient C_9 is given by

$$\begin{aligned}
C_9 = {} & (I_z-I_x)(I_z-I_y)\Omega_3^2\mu_{d1}^2\mu_{b13}^2 + 2b_1\mu_{b13}^2\Omega_3^3S_{23}^1(I_x-I_z) \\
& + 2\Omega_3^3(\tau_{d2}/\tau_{d1})f_2\mu_{b13}^2[I_xS_{13}^1-S_{21}^4(I_z-I_x)] \\
& + \Omega_3^2\{\Omega_3\mu_{b13}^7\mu_{d1}^3[S_{23}^4(I_z-I_x) - S_{31}^3I_x] - I_xS_{31}^3(\mu_{b13}^5\mu_{d1}^3+\Omega_3\mu_{b13}^7\mu_{d1}^2)\} \\
& - \Omega_3^2\{\mu_{d1}^5\mu_{b13}^2S_{13}^1(I_z-I_y) + \mu_{d1}^7\mu_{b13}^2[\Omega_3I_yS_{23}^1 + S_{11}^4(I_z-I_y)]\} \\
& + \Omega_3^3S_{31}^3(I_z-I_y)(\mu_{d1}^3\mu_{b13}^4+\Omega_3\mu_{b13}^6\mu_{d1}^2) - \Omega_3^3\mu_{d1}^3\mu_{b13}^6[I_yS_{32}^3+S_{13}^4(I_z-I_y)] \\
& + \Omega_3^5[(\mu_{d1}^1\mu_{b13}^7+\mu_{b13}^5\mu_{d1}^7-\mu_{b13}^4\mu_{d1}^8-\mu_{b13}^6\mu_{d1}^6)(S_{23}^4S_{11}^4-S_{21}^4S_{13}^4) \\
& + (\mu_{d1}^7\mu_{b13}^7-\mu_{b13}^6\mu_{d1}^8)(S_{23}^4S_{13}^1+S_{32}^3S_{11}^4-S_{13}^4S_{23}^1+S_{21}^4S_{31}^3)].
\end{aligned} \qquad (80)$$

Expressions for the other coefficients in Eq. (79) are similarly obtained.

For the roots of α in Eq. (79) to be purely imaginary,

$$C_5 = C_7 = C_9 = 0 \qquad (81)$$

Examining Eq. (80) and similar expressions for C_5 and C_7 it becomes evident that Eq. (81) can be satisfied for arbitrary values of τ_{bij} and τ_{di} if and only if

$$\Omega_3 = \mu_{d1}^2 = 0 \qquad (82)$$

Equation (82) is another proof of our previous claim that stochastic stability is possible only for undamped three-axes stabilized satellites.

THREE-AXES STABILIZED SATELLITES

For a three-axes stabilized satellite, the constraints given by Eq. (78) are almost always satisfied. Also for this configuration,

$$C_5 = C_7 = C_8 = C_9 = C_{10} = 0 \qquad (83)$$

such that Eq. (79) becomes

$$\alpha^4[C_6+C_4\alpha^2] = 0 \tag{84}$$

Hence the required stability criteria are

$$\begin{aligned} 0 \leq C_4 = {} & I_x I_y \mu^1_{d1} \mu^1_{b13} + 2b_1 \mu^1_{b13} I_x S^4_{21} + \mu^1_{d1} I_x S^4_{23} + I_y S^4_{11} \mu^5_{d1} \mu^1_{b13} \\ & + I_y \mu^1_{d1} \mu^4_{b13} S^4_{13} + (\mu^5_{d1}\mu^5_{b13} - \mu^6_{d1}\mu^4_{b13})(S^4_{11}S^4_{23} - S^4_{21}S^4_{13}) \end{aligned} \tag{85}$$

and

$$\begin{aligned} 0 \leq C_6 = {} & I_y S^4_{13} \mu^3_{d1} \mu^4_{b13} + I_y S^4_{11} \mu^5_{d1} \mu^2_{b13} + I_x S^4_{23} \mu^5_{b13} \mu^3_{d1} \\ & + 2b_1 \mu^2_{b13} I_x S^4_{21} + I_x I_y (\mu^3_{d1} \mu^1_{b13} + \mu^1_{d1} \mu^2_{b13}) \quad . \end{aligned} \tag{86}$$

Constraints given by Eqs. (85) and (86) can be satisfied usually without great difficulty, irrespective of the values of τ_{bij} and τ_{di}. This is due to the fact that C_4 and C_6 are mainly the mass and stiffness terms of the satellite model. Hence, it can be concluded that three-axes stabilized satellites are more likely to be stable under random environmental and control torques.

ACKNOWLEDGEMENTS

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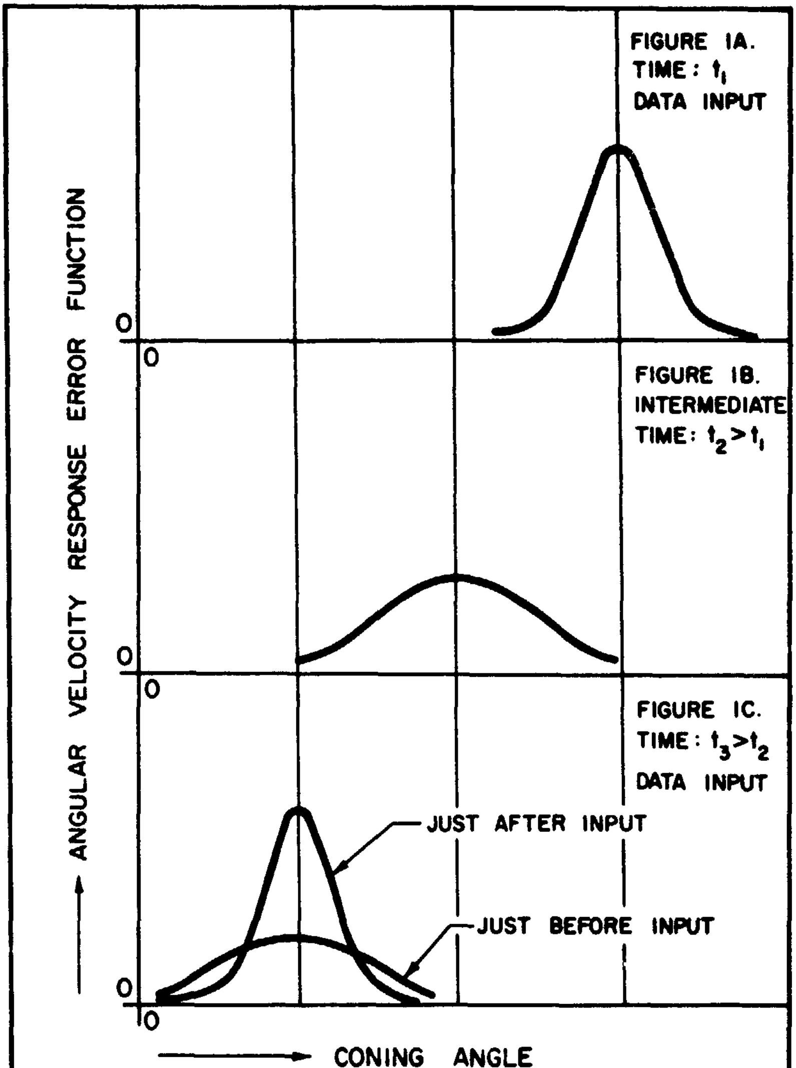

FIGURE 1: INFORMATION STABILITY

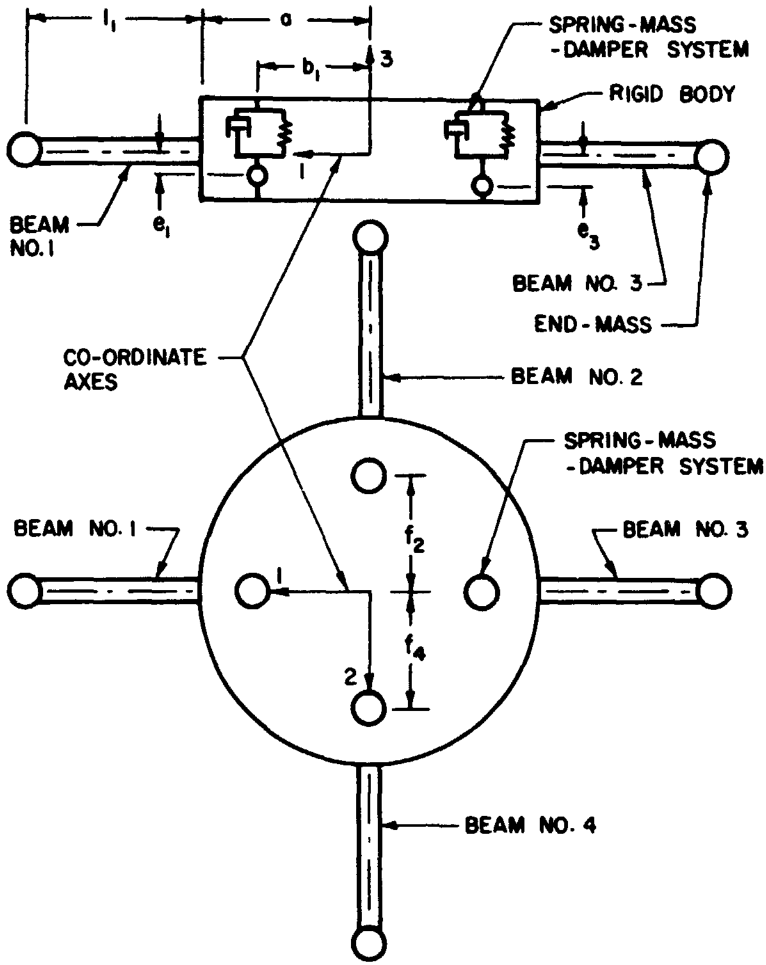

FIGURE 2: SATELLITE CONFIGURATION

STABILITY AND CONTROL OF FLEXIBLE SATELLITES:

PART II - CONTROL

T. C. Huang and Aniruddha Das

ABSTRACT

This is the second part of an earlier investigation. In this section, it is demonstrated that, by monitoring the deformations of the flexible elements of a satellite, the effectiveness of the satellite control system can be increased considerably. A simple model of a flexible satellite had been analyzed in the first part of this work. The same model has been used here for digital computer simulations.

NOMENCLATURE

$[A_i']$, $i = 1\text{-}5$	= Matrices governing the equations of motion of flexible structural elements of the satellite; Eq. (1)
$[B_i']$, $i = 1,2$	= Matrices governing the satellite motion; Eqs. (3,7,8).
$[\hat{B}(t)]$	= Upper (3x3) left corner submatrix of $[\Phi(t)][B_1']^{-1}$.
$\underline{f}$	= External forcing function; Eq. (6).
$\underline{f}^*$	= External torque vector on the satellite; Eq. (2).
$[I]$	= Identity matrix.
$[K]$	= System fundamental matrix; Eq. (22).
$[K_1]$	= Matrix defined by Eq. (28).
n	= Number of scalar elements in $\underline{q}'$.
$[0]$	= Null matrix.
$[P_i']$, $i = 1\text{-}5$	= Matrices governing the rotational motion of the satellite; Eq. (2)
q', $\{q_i'\}$	= Generalized structural position coordinate vector.
$\underline{q}_{bi}$	= Generalized position coordinate for the ith beam.
T	= Terminal time for optimal control.
t	= Time.

U	= Torque magnitude parameter, defined by Eq. (25).
$\underline{u}$, $\{u_i\}$	= Generalized control vector; Eqs. (5), (22).
$\underline{u}^*$	= Control torque vector on the satellite; Eq. (2).
$\underline{u}^i$, $\{u_j^i\}$	= Various control torque functions; Eqs. (14) - (21).
$\underline{x}$	= State vector; Eqs. (3), (4), (22).
$\underline{y}$	= Uncontrolled response; Eq. (22).
$\underline{y}_{di}$	= Position vector of the ith spring-mass-damper system.
$\underline{y}_{ri}$	= Position vector of the ith beam-end mass.
$\underline{\lambda}^*$, $\{\lambda_i^*\}$	= Control system parameter; Eqs. (24), (26).
$\underline{\mu}^*$, $\{\mu_i^*\}$	= Relative control torque magnitude vector; Eqs. (24), (27).
τ	= Dummy time variable.
$[\Phi]$	= Fundamental matrix of $-[B_1']^{-1}[B_2']$; Eqs. (9), (10).
$[\Phi_i]$, $i = 1-4$	= Component matrices of $[\Phi]$; Eq. (11).
$\underline{\omega}$, $\{\omega_i\}$	= Angular velocity vector of the satellite; Eqs. (1), (2).
$\underline{\omega}^i$, $\{\omega_j^i\}$	= Various simulation responses of $\underline{\omega}$; Eqs. (14) - (21).

INTRODUCTION

In the first part [1] of this study, the question of stochastic stability of flexible satellites was discussed. Specific stability criteria were developed for a simple flexible model of a satellite (shown in Figure 1). In this part of the study, we determine whether it is possible to increase the pointing accuracy of a satellite by observing the deflections of the flexible elements. To do this, we use the same satellite configuration (Figure 1) and the theoretical model developed in Ref. [1].

Likins and Fleischer [2] have shown that the flexible elements of spacecraft can have a destabilizing influence. They have shown a method of designing a proportional linear control system employing root-locus plots and eigenvalue analyses. The control loop gains in [2] were based on a dynamic model, using hybrid coordinates, of a spacecraft containing long flexible beams. An essentially similar approach was employed by DiLorenzo and Santinelli [3]. Here also a linear proportional control system was designed by considering the equations of motion of the spacecraft along with those of the flexible elements. The spacecraft model in [3] consisted of a rigid body with two spring-mass systems.

In this study, a time-optimal 'bang bang' control policy has been assumed. The method of calculating the control torques is essentially the same as that given in Ref. [4]. Full details of the computation of control torques are presented in Ref. [5]. Apart from the control policy, this analysis differs from Refs. [2,3] in another important aspect. In the analyses of Refs. [2,3], the deflections of the flexible elements are not observed. Hence, zero initial deflections and velocities of the flexible elements are inherently assumed. The present method can accommodate arbitrarily large initial conditions of the flexible elements of the satellite.

THEORETICAL BASIS OF COMPARISON

The theoretical analysis and comparison of the satellite responses is based on the dynamic model explained in Section 6 of Ref. [1]. It was shown there [1] that, by using the Galerkin's method, the deflections of the flexible elements of the satellite are governed by purely time-dependent generalized position vectors, $\underline{q}_{bi}(t)$, $\underline{y}_{ri}(t)$ and $\underline{y}_{di}(t)$. It was also shown that these vectors can be condensed subsequently, and reduced to a vector $\underline{q}'(t)$ by applying suitable boundary and continuity conditions. Usually the number of elements in $\underline{q}'$ is much smaller than that in the set $[\underline{q}_{bi}, \underline{y}_{ri}, \underline{y}_{di}]^T$.

Let $\underline{\omega}(t)$ be the angular velocity vector of the satellite. Let $\underline{u}^*(t)$ and $\underline{f}^*(t)$ be the control torque and environmental torque vector on the satellite. Given these definitions, it is well known [1,6,7] that the satellite response is governed by a pair of matrix equations of the form

$$[A_1']\ddot{\underline{q}}'(t) + [A_2'(\underline{\omega},t)]\dot{\underline{q}}'(t) + [A_3'(\underline{\omega},t)]\underline{q}'(t)$$

$$= [A_4']\dot{\underline{\omega}}(t) + [A_5'(\underline{\omega})]\underline{\omega}(t) \tag{1}$$

and

$$[P_1']\ddot{\underline{q}}(t) + [P_2'(\underline{\omega},t)]\dot{\underline{q}}'(t) + [P_3'(\underline{\omega},t)]\underline{q}'(t)$$

$$= [P_4']\dot{\underline{\omega}}(t) + [P_5'(\underline{\omega})]\underline{\omega}(t) + \underline{u}^*(t) + \underline{f}^*(t). \tag{2}$$

Equation (1) governs the flexible motion of the beams, spring-mass-dampers, and beam-end masses of the satellite model. Equation (2) is based on the principle of conservation of angular momentum of the satellite. If $\underline{q}'(t)$ is a (nx1) vector, then there are 'n' scalar equations in Eq. (1). Equation (2) always has three scalar equations. Equations (1) and (2) correspond to Eqs. (44) and (45) of Ref. [1].

Equations (1) and (2) are now combined together to form one first order equation given by

$$[B_1']\dot{\underline{x}}(t) + [B_2']\underline{x}(t) = \underline{u}(t) + \underline{f}(t) \tag{3}$$

where

$$\underline{x} = [\underline{\omega}, \dot{\underline{q}}', \underline{q}']^T \tag{4}$$

$$\underline{u} = [\underline{u}^*,\underline{0},\underline{0}]^T \tag{5}$$

$$\underline{f} = [\underline{f}^*,\underline{0},\underline{0}]^T \tag{6}$$

$$[B_1'] = \begin{bmatrix} [P_4'] & -[P_1'] & [0] \\ [A_4'] & -[A_1'] & [0] \\ [0] & [0] & [I] \end{bmatrix} \tag{7}$$

and

$$[B_2'] = \begin{bmatrix} [P_5'] & -[P_2'] & -[P_3'] \\ [A_5'] & -[A_2'] & -[A_3'] \\ [0] & -[I] & [0] \end{bmatrix} \tag{8}$$

Let $[\Phi(t)]$ be the fundamental matrix of the homogeneous equation

$$\dot{\underline{x}} = -[B_1']^{-1}[B_2']\underline{x} \tag{9}$$

such that the solution of Eq. (3) is given by

$$\underline{x}(t) = [\Phi(t)]\underline{x}(0) + \int_0^t [\Phi(t-\tau)][B_1']^{-1}[\underline{u}(\tau) + \underline{f}(\tau)]d\tau \ . \tag{10}$$

Let $[\Phi(t)]$ be composed of $[\Phi_1(t)]$, $[\Phi_2(t)]$, $[\Phi_3(t)]$ and $[\Phi_4(t)]$ such that

$$[\Phi] = \begin{bmatrix} \begin{matrix} [\Phi_1] & [\Phi_2] & [\Phi_3] \\ (3x3) & (3xn) & (3xn) \end{matrix} \\ [\Phi_4] \\ [2n \ x(2n+3)] \end{bmatrix} \tag{11}$$

when $\underline{\omega}(t)$ and $\underline{q}'(t)$ are (3x1) and (nx1) vectors, respectively. Then the equations corresponding to $\underline{\omega}(t)$ can be separated from Eq. (10) in the form

$$\underline{\omega}(t) = [\Phi_1(t)]\underline{\omega}(0) + [\Phi_2(t)]\dot{\underline{q}}'(0) + [\Phi_3(t)]\underline{q}'(0)$$
$$+ \int_0^t [\hat{B}(t-\tau)][\underline{u}^*(\tau) + \underline{f}^*(\tau)]d\tau \tag{12}$$

where $[\hat{B}(t)]$ is the (3x3) upper left hand corner submatrix of $[\Phi(t)][B_1']^{-1}$.

It should be noted that previous investigations [2,3] were concerned mainly with the determination of $[\Phi_1(t)]$ and $[\hat{B}(t)]$ and then with the approximation of Eq. (12) by

$\dot{q}_1'(0) = \dot{q}_2'(0) = 0.01$; $\dot{q}_j'(0) = 0$, $j \neq 1,2$.

The complete numerical experiment is performed through the following steps:

Step 1: A time interval [0,T] in which the controls are to be effected is fixed. In this case T was taken as 5.0 secs.

Step 2: The satellite is assumed to be rigid, and without controls, such that $\underline{\omega}(t)$ is given by the solution $\underline{\omega}^1(t)$, of the equation

$$[P_4']\dot{\underline{\omega}}^1(t) + [P_5']\underline{\omega}^1(t) + \underline{f}^*(t) = \underline{0} . \qquad (14)$$

Equation (14) is integrated and the responses $\omega_1^1(t)$ and $\omega_2^1(t)$ are plotted in Figure 2.

Step 3: The satellite is assumed to be rigid and subjected to a time-optimal 'bang-bang' control, $\underline{u}^1(t)$, such that $\underline{\omega}(t)$ is given by the solution $\underline{\omega}^2(t)$, of the equation

$$[P_4']\dot{\underline{\omega}}^2(t) + [P_5']\underline{\omega}^2(t) + \underline{u}^1(t) + \underline{f}^*(t) = \underline{0} . \qquad (15)$$

The $\underline{u}^1(t)$ are computed so as to yield $\underline{\omega}^2(T) = 0$ by the method shown in Appendix A. Equation (15) is integrated and the responses $\omega_1^2(t)$ and $\omega_2^2(t)$ are plotted in Figure 3.

Step 4: The satellite is assumed to be flexible, without control and with $\dot{\underline{q}}'(0) = \underline{q}'(0) = \underline{0}$, such that $\underline{\omega}(t)$ is given by $\underline{\omega}^3(t)$. Here

$$\underline{\omega}^3(t) = [\Phi_1(t)]\underline{\omega}(0) + \int_0^t [\hat{B}(t-\tau)]\underline{f}^*(\tau)d\tau . \qquad (16)$$

The responses $\omega_1^3(t)$ and $\omega_2^3(t)$ from Eq. (16) are plotted in Figure 4.

Step 5: The satellite is assumed to be flexible, with $\dot{\underline{q}}'(0) = \underline{q}'(0) = \underline{0}$. The satellite is subjected to the control torque $\underline{u}^1(t)$ computed in Step 3, such that $\underline{\omega}(t)$ is given by $\underline{\omega}^4(t)$, where

$$\underline{\omega}^4(t) = [\Phi_1(t)]\underline{\omega}(0) + \int_0^t [\hat{B}(t-\tau)][\underline{f}^*(\tau) + \underline{u}^1(\tau)]d\tau . \qquad (17)$$

The responses $\omega_1^4(t)$ and $\omega_2^4(t)$ from Eq. (17) are plotted in Figure 5.

Step 6: The satellite is assumed to be flexible, with $\dot{\underline{q}}'(0) = \underline{q}'(0) = \underline{0}$, and subjected to a time-optimal 'bang-bang' control, $\underline{u}^2(t)$, such that $\underline{\omega}(t)$ is given by $\underline{\omega}^5(t)$, where

$$\underline{\omega}^5(t) = [\Phi_1(t)]\underline{\omega}(0) + \int_0^t [\hat{B}(t-\tau)][\underline{f}^*(\tau) + \underline{u}^2(\tau)]d\tau . \qquad (18)$$

The $\underline{u}^2(t)$ are computed so as to yield $\underline{\omega}^5(T) = \underline{0}$ by the method shown in Appendix A. The responses $\omega_1^5(t)$ and $\omega_2^5(t)$ from Eq. (18) are plotted in Figure 6.

Step 7: The satellite is assumed to be flexible, with $\dot{\underline{q}}'(0) \neq 0 \neq \underline{q}'(0)$ and without control, such that $\underline{\omega}(t)$ is given by $\underline{\omega}^6(t)$, where

$$\underline{\omega}^6(t) = [\Phi_1(t)]\underline{\omega}(0) + [\Phi_2(t)]\dot{\underline{q}}'(0) + [\Phi_3(t)]\underline{q}'(0)$$
$$+ \int_0^t [\hat{B}(t-\tau)][\underline{f}^*(\tau)]d\tau \ . \qquad (19)$$

The responses $\omega_1^6(t)$ and $\omega_2^6(t)$ are plotted in Figure 7.

Step 8: The satellite is assumed to be flexible, with $\dot{\underline{q}}'(0) \neq \underline{0} \neq \underline{q}'(0)$ and subjected to the control torque $\underline{u}^2(t)$ computed in Step 6, such that $\underline{\omega}(t)$ is given by $\underline{\omega}^7(t)$, where

$$\underline{\omega}^7(t) = [\Phi_1(t)]\underline{\omega}(0) + [\Phi_2(t)]\dot{\underline{q}}'(0) + [\Phi_3(t)]\underline{q}'(0)$$
$$+ \int_0^t [\hat{B}(t-\tau)][\underline{u}^2(\tau) + \underline{f}^*(\tau)]d\tau \ . \qquad (20)$$

The responses $\omega_1^7(t)$ and $\omega_2^7(t)$ are plotted in Figure 8.

Step 9: The satellite is assumed to be flexible, with $\dot{\underline{q}}'(0) \neq 0 \neq \underline{q}'(0)$. It is also subjected to a time-optimal 'bang-bang' control, $\underline{u}^3(t)$, such that $\underline{\omega}(t)$ is given by $\underline{\omega}^8(t)$, where

$$\underline{\omega}^8(t) = [\Phi_1(t)]\underline{\omega}(0) + [\Phi_2(t)]\dot{\underline{q}}'(0) + [\Phi_3(t)]\underline{q}'(0)$$
$$+ \int_0^t [\hat{B}(t-\tau)][\underline{u}^3(\tau) + \underline{f}^*(\tau)]d\tau \ . \qquad (21)$$

The torques $\underline{u}^3(t)$ are also computed to yield $\underline{\omega}^8(T) = \underline{0}$ by the method shown in Appendix A. The responses $\omega_1^8(t)$ and $\omega_1^9(t)$ from Eq. (21) are plotted in Figure 9.

COMPARISON AND EVALUATION

One important result of the simulation, as seen from Figures 2 and 3, is that the control sequence $\underline{u}^1(t)$ is very effective on the rigid model of the satellite. But Figure 5 shows that, for the same values of $\underline{\omega}(0)$, $\underline{u}^1(t)$ produces unwanted non-zero values of $\underline{\omega}(T)$ when it is applied to the flexible satellite model, although $\dot{\underline{q}}(0)$ and $\underline{q}(0)$ are assumed to be zero. Thus, another important result, presented in Figures 5 and 6, shows that $\underline{u}^2(t)$ is more effective than $\underline{u}^1(t)$ when a flexible satellite model is considered. Up to this point, then, we have essentially the same conclusion as that in Refs. [2], [3], that for a flexible satellite the control should not be based on a rigid model. The difference between Refs. [2], [3] and the present study is in the adopted control policy. 'Bang-bang' controls have been used here instead of linear proportional control.

The most important results are presented in Figures 8 and 9. When the $\dot{\underline{q}}(0)$ and $\underline{q}(0)$ are observed and found different from zero, $u^2(t)$ does not lead to the required zero values of $\underline{\omega}(T)$. In contrast, $\underline{u}^3(t)$, which is based on the observed values of $\dot{\underline{q}}(0)$ and $\underline{q}(0)$, yields zero values of $\underline{\omega}(T)$. Another point to be considered is the divergence of $\underline{\omega}(t)$ from zero in the two cases. The maximum divergence of $\underline{\omega}(t)$ and $\underline{u}^2(t)$ is 11.0×10^{-4} rads/sec, while that with $\underline{u}^3(t)$ is only 7.0×10^{-4} rads/sec. This bears out the theoretical claims that a control based on Eq. (12) is more effective than one based on Eq. (13) and that the effectiveness of a control system can be greatly improved if the deflections of the flexible elements of a satellite are observed.

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APPENDIX A

The method of computing the time-optimal control torques for a system given by

$$\underline{x}(t) = \underline{y}(t) + \int_0^t [K(t-\tau)]\underline{u}(\tau)d\tau \tag{22}$$

is now presented. Reference [5] presents computing algorithms and other details of the method. In Eq. (22), $\underline{x}(t)$ is the output vector of the system, $\underline{u}(t)$ is the control vector, and $\underline{y}(t)$ and $[K(t)]$ are known vector and matrix functions of the time, t.

It is assumed that, for a given t = T, $\underline{u}(t)$ should be such that

$$\underline{x}(T) = \underline{0} \tag{23}$$

and $|\underline{u}(t)|$ for all t is a minimum. Thus, the minimum time problem is converted to the equivalent minimum control effort problem. The solution for $\underline{u}(t)$ is then given by [5].

$$u_j(t) = U(T)\mu_j^* \operatorname{sgn}[\sum_i \lambda_i^* K_{ij}(T-t)] \tag{24}$$

where

$$U(T) = 1.0/[\min_{\lambda_i^*} \sum_j \int_0^T |\sum_i \lambda_i^* K_{ij}(T-\tau)| d\tau] \tag{25}$$

such that

$$\sum_i \lambda_i^* y_i(T) = 1.0 \tag{26}$$

and

$$\mu^* = -\frac{1}{U}[K_1]^{-1}\underline{y}(T) \tag{27}$$

$$[K_1]_{ij} = \int_0^T [K_{ij}(t-\tau)] \operatorname{sgn}[\sum_r \lambda_r^* K_{rj}(T-\tau)]d\tau \tag{28}$$

The summation convention of repeated indices is not to be used in Eqs. (24) to (27) above.

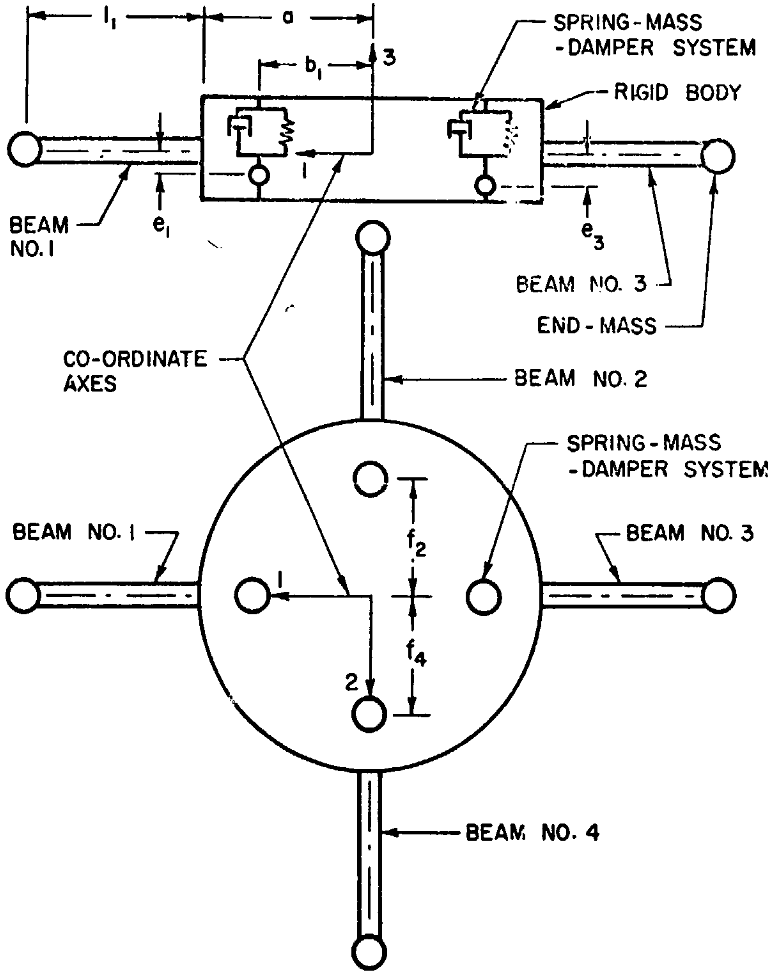


FIGURE 1: SATELLITE CONFIGURATION

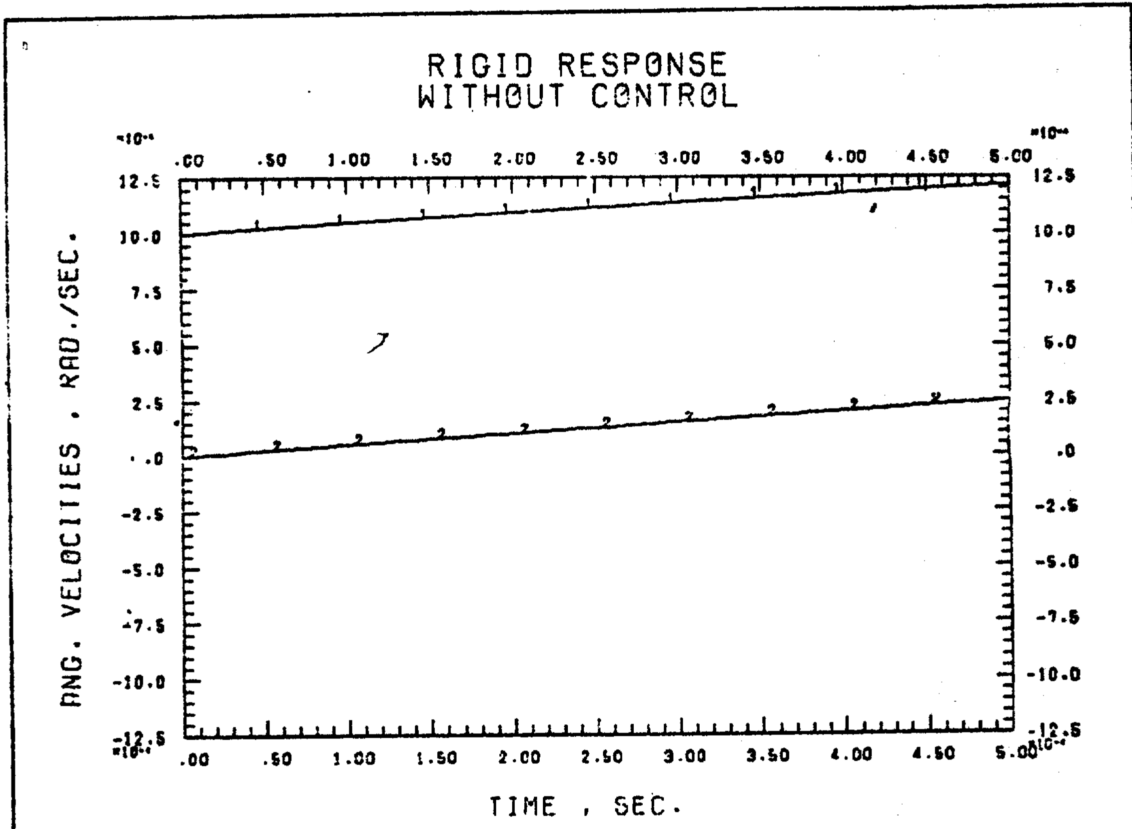

RIGID RESPONSE
WITHOUT CONTROL
ANG. VELOCITIES , RAD./SEC.
TIME , SEC.
$*10^{-4}$
.00 .50 1.00 1.50 2.00 2.50 3.00 3.50 4.00 4.50 5.00
12.5 10.0 7.5 5.0 2.5 .0 -2.5 -5.0 -7.5 -10.0 -12.5

Figure 2

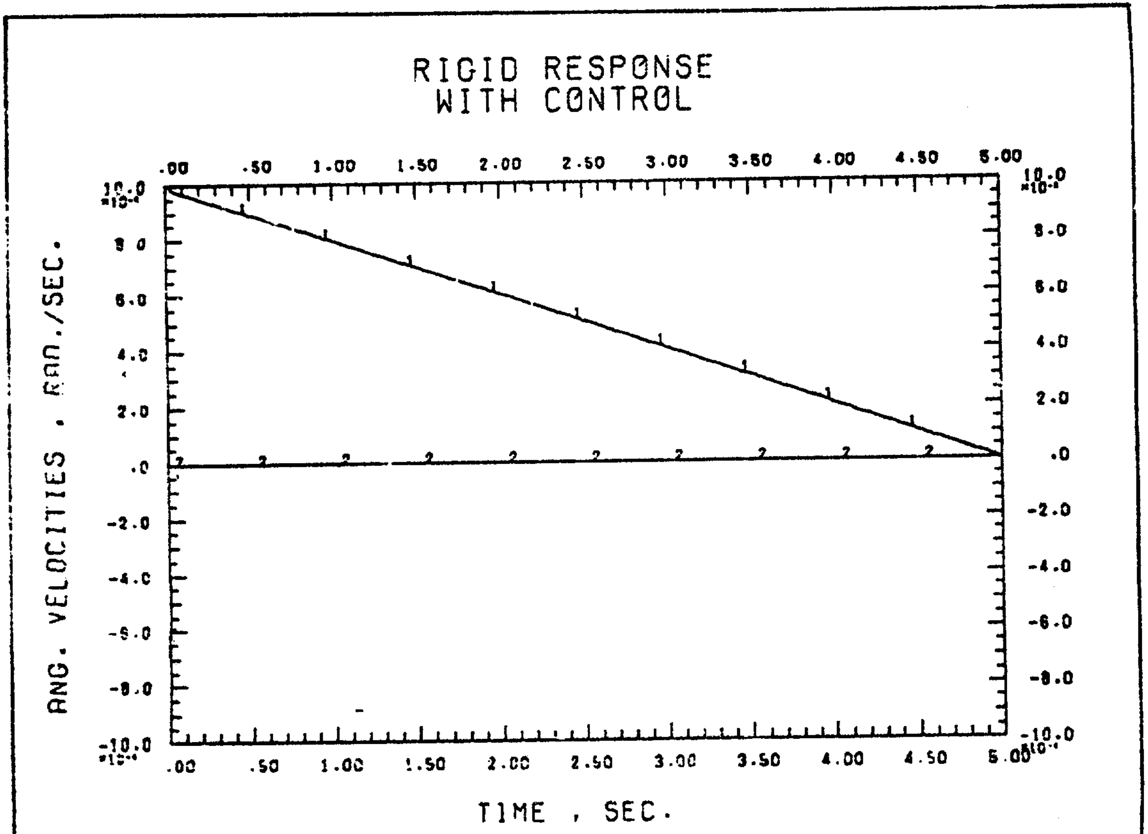

RIGID RESPONSE
WITH CONTROL
ANG. VELOCITIES , RAD./SEC.
TIME , SEC.
.00
.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
10.0
8.0
6.0
4.0
2.0
.0
-2.0
-4.0
-6.0
-8.0
-10.0

Figure 3

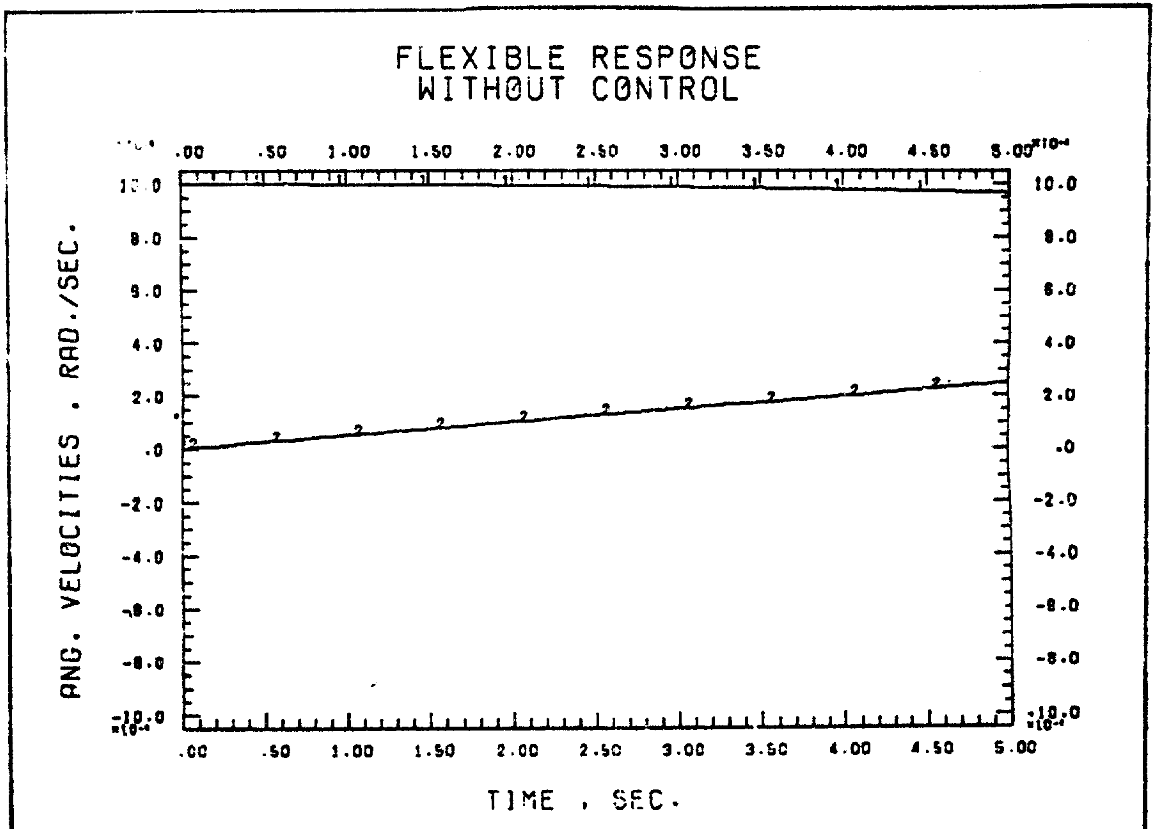

FLEXIBLE RESPONSE
WITHOUT CONTROL
ANG. VELOCITIES . RAD./SEC.
TIME , SEC.

Figure 4

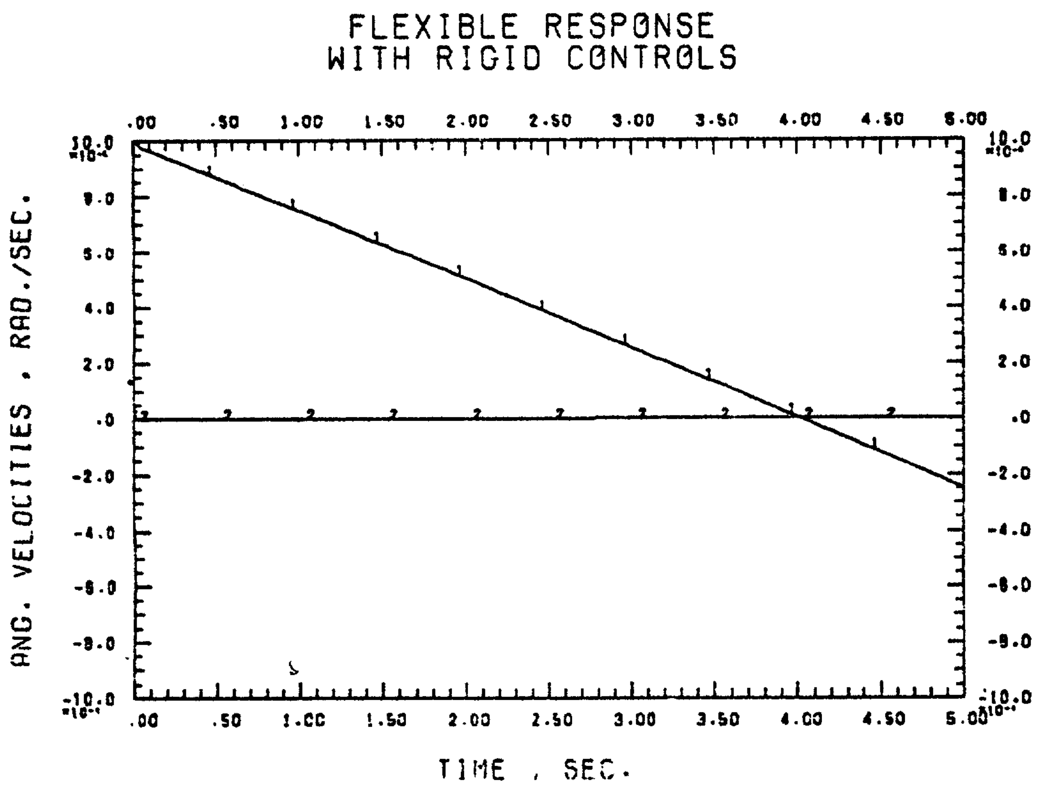

FLEXIBLE RESPONSE
WITH RIGID CONTROLS
ANG. VELOCITIES , RAD./SEC.
TIME , SEC.

Figure 5

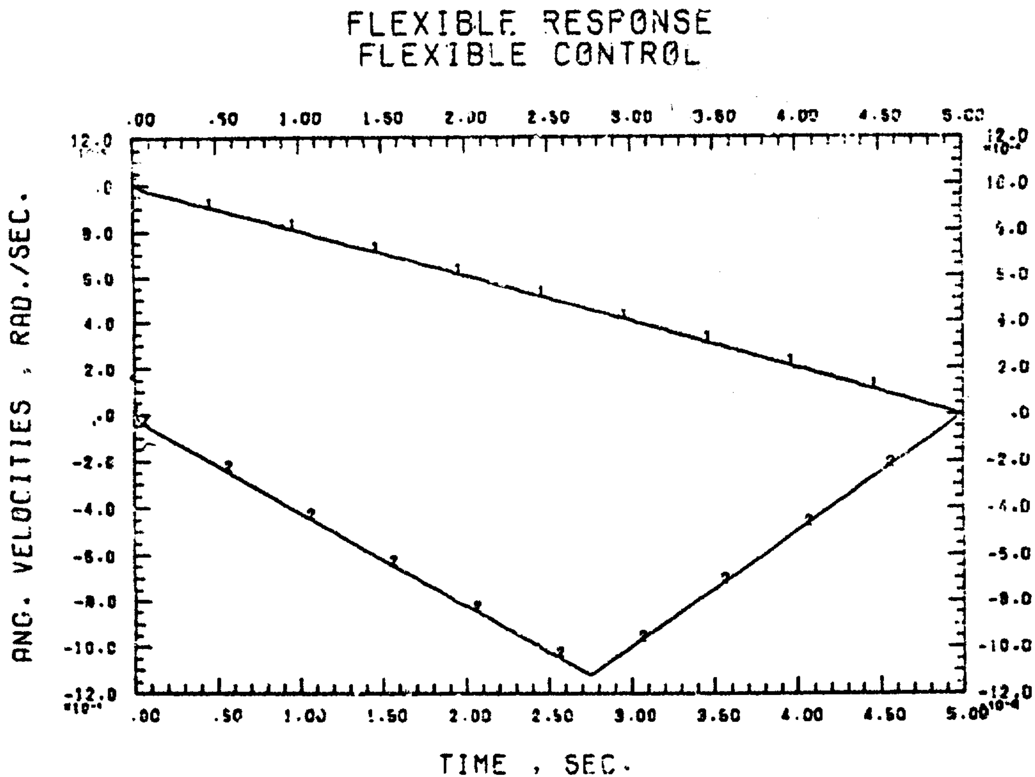

FLEXIBLE RESPONSE
FLEXIBLE CONTROL
ANG. VELOCITIES , RAD./SEC.
TIME , SEC.

Figure 6

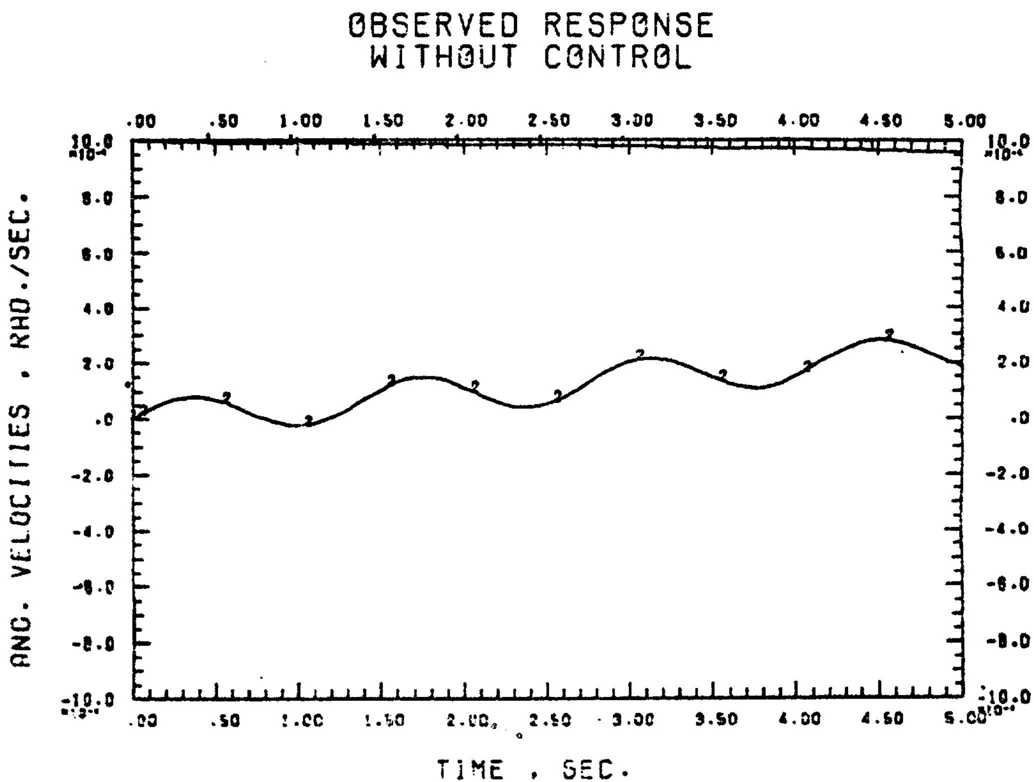
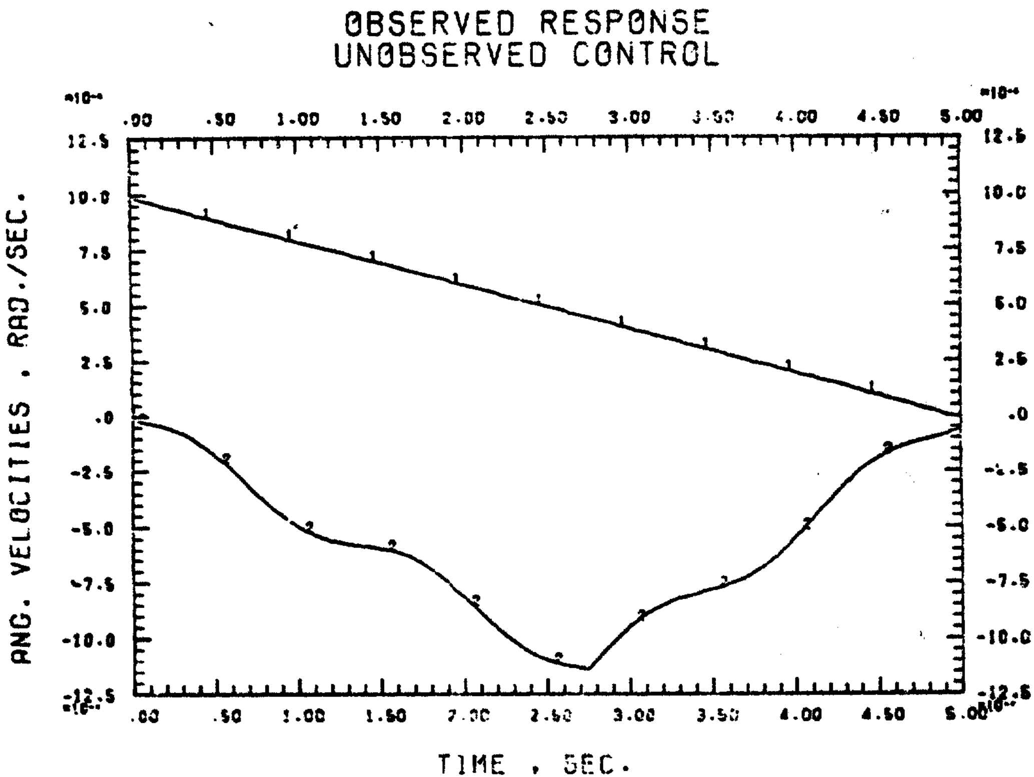

OBSERVED RESPONSE
WITHOUT CONTROL
ANG. VELOCITIES , RAD./SEC.
TIME , SEC.
.00 .50 1.00 1.50 2.00 2.50 3.00 3.50 4.00 4.50 5.00
10.0 8.0 6.0 4.0 2.0 .0 -2.0 -4.0 -6.0 -8.0 -10.0
*10⁻⁴

Figure 7



OBSERVED RESPONSE
UNOBSERVED CONTROL
ANG. VELOCITIES , RAD./SEC.
TIME , SEC.

Figure 8

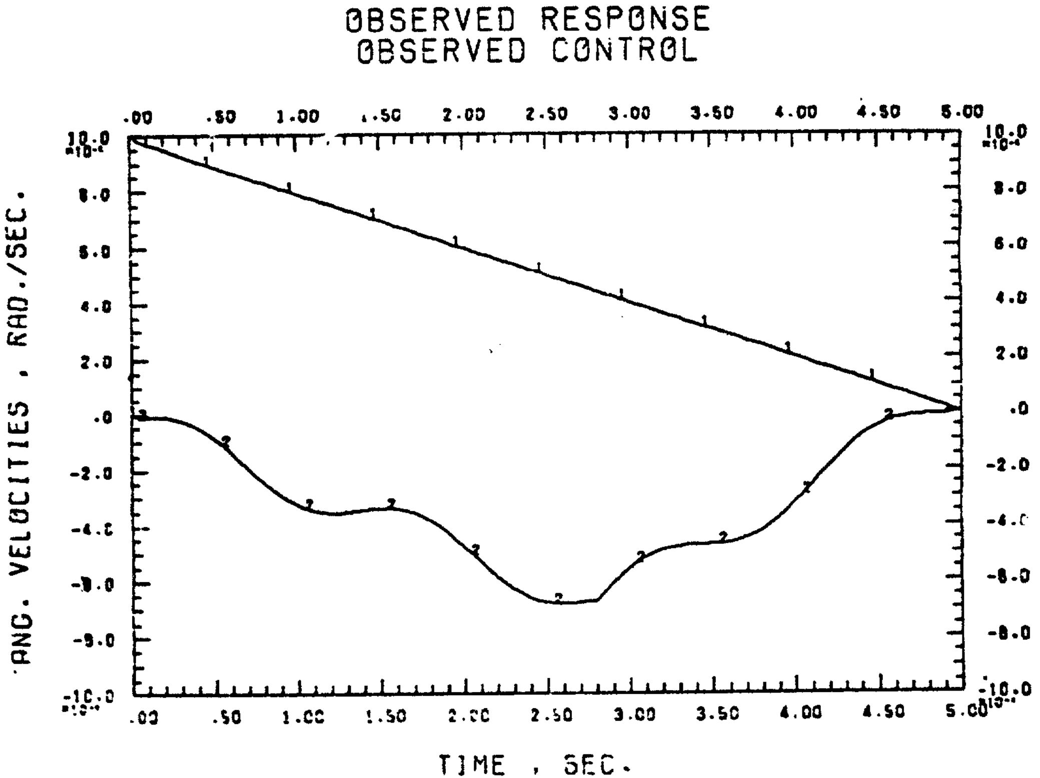

OBSERVED RESPONSE
OBSERVED CONTROL
ANG. VELOCITIES , RAD./SEC.
TIME , SEC.

Figure 9