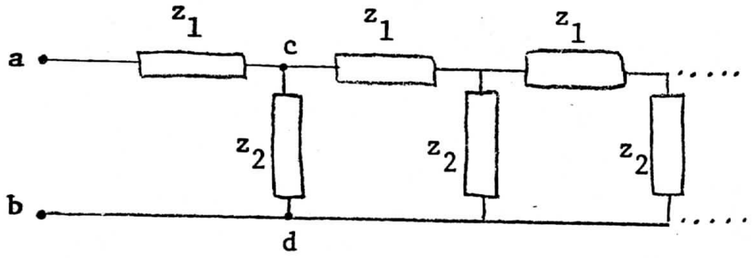
# Notes on Filters

UW/SSEC
Paul Menzel
April 1976

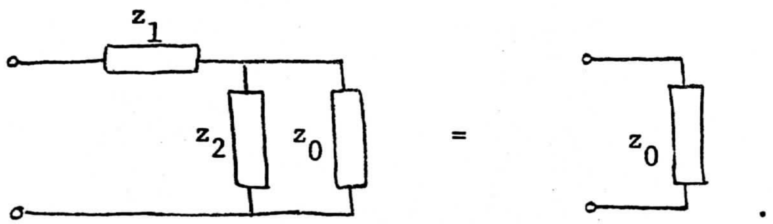
## Introduction to Filters

### A. A Ladder Network

Consider the infinite ladder and let the impedance across ab be




represented by $z_0$. To determine $z_0$ we notice that this infinite network is unchanged if one more section is added to the front end, thus



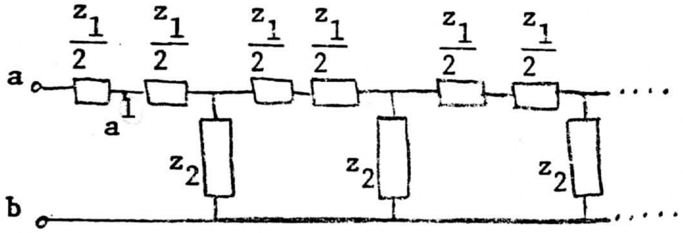

Therefore we can write for the characteristic impedance

$$z_0 = z_1 + \cfrac{1}{\cfrac{1}{z_2} + \cfrac{1}{z_0}}$$

or

$$z_0 = \frac{z_1}{2} + \sqrt{\frac{z_1^2}{4} + z_1 z_2} \;;$$

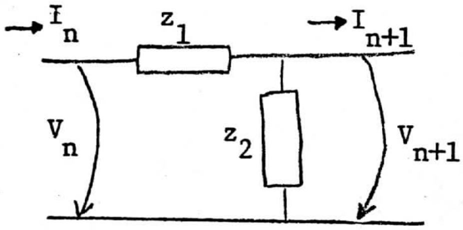
if $z_1 = iwL$ and $z_2 = 1/iwC$ then $z_0 = \frac{iwL}{2} + \sqrt{\frac{L}{C} - \frac{w^2L^2}{4}}$. The first term $\frac{z_1}{2}$ is just one half the impedance of the first element. It is simpler to redraw our infinite network as follows



Then the impedance across a b, $z_0 = \frac{L}{C} - \frac{w^2L^2}{4}$. Now there are two interesting cases: (1) if $w^2 < \frac{4}{LC}$ then $z_0$ is real and continuous absorption of energy occurs (ie. the current penetrates far down the line) or (2) if $w^2 > \frac{4}{LC}$ then $z_0$ is imaginary and we see no such propagation.

B. Filters

Suppose we hook up an AC generator across ab and wish to determine the voltage across the $n^{th}$ section of the infinite ladder.




$$V_n - V_{n+1} = I_n z_1 = V_n \frac{z_1}{z_0}$$

or

$$\frac{V_{n+1}}{V_n} = 1 - \frac{z_1}{z_0} = \alpha = \text{propagation factor.}$$

The voltage after the $n^{th}$ section is

$$V_n = \alpha^n (\Sigma).$$

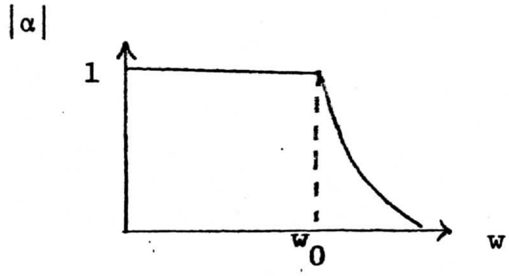
($\Sigma$ = AC voltage across ab.)

For our LC ladder

$$\alpha = \frac{\sqrt{L/C - w^2L^2/4} - i\,wL/2}{\sqrt{L/C - w^2L^2/4} + i\,wL/2}$$

$$= \begin{cases} e^{id} \text{ for } w < w_{cutoff} = \sqrt{4/LC} \\ < 1 \text{ for } w > w_{cutoff} \end{cases}$$

For $w < w_C$ the magnitude of the voltage across each section is the same only the phase changes; for $w > w_C$ the voltage decreases by the factor $\alpha$ which is less than 1. Thus we have a low pass filter with the following propagation factor.




If we interchange L's and C's then we have a high pass filter. Whatever happens at w for low pass occurs at $\frac{1}{w}$ for high pass. To translate this behavior to a finite network end it off with an impedance equal to the characteristic impedance $z_0$. In practice it is not possible to exactly reproduce $z_0$ for all frequencies, but it is often possible to do so for a certain range of frequencies. For example the L-C ladder tied off with a resistance $R = \sqrt{L/C}$ behaves much as we have described the infinite L-C ladder.

Different Types of Filters

C. Response Functions

A response function is defined as the ratio of the output voltage to the input voltage. All response functions are ratios of polynomials in $s = iw$ with real coefficients

$$R(s) = \frac{a_0 s^p + a_1 s^{p-1} + \ldots + a_p}{b_0 s^q + b_1 s^{q-1} + \ldots + b_q}$$

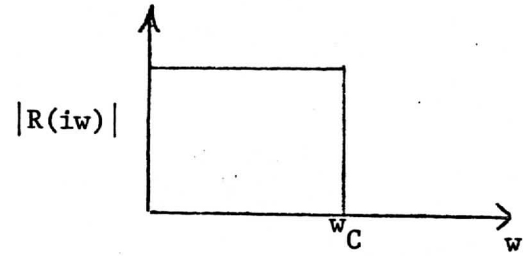
$$= A \frac{(s - p_1)(s - p_2)\ldots(s - p_p)}{(s - q_1)(s - q_2)\ldots(s - q_q)}$$

zeroes ($p$'s), poles ($q$'s)

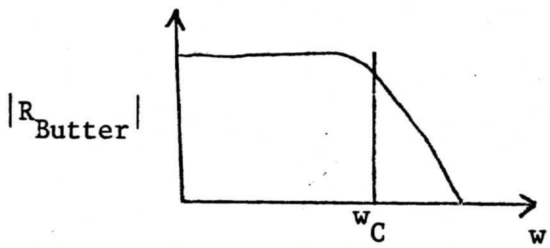
Any response function can be expressed uniquely in terms of its poles and zeros to within a constant factor. For a transfer function of the above form there are p + q + 1 independent coefficients which are used to specify component values in the filter.

D. Different Tradeoffs in Filter Design

To achieve as close to possible an ideal frequency response

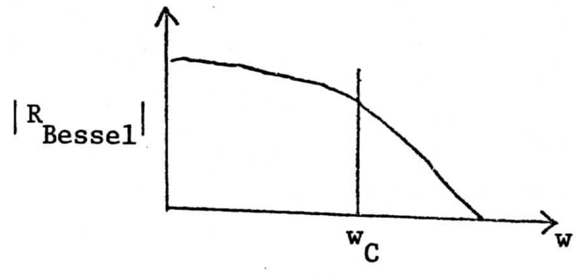



various tradeoffs come into consideration. One common criterion, called the maximally flat criterion, is that as many of the derivatives of $|R(iw)|$ at $w = 0$ as possible should be equal to zero. Application of this criterion leads to Butterworth filters, which have monotonically increasing attentuation, good phase characteristics and low passband losses, but the cutoff is not sharp.

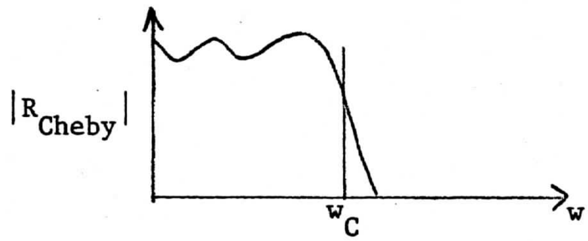



The maximally flat phase criterion requires that derivatives of the phase angle of $R(iw)$ at $w = 0$ beyond the first derivative should be zero ($1^{st}$ derivative of $\phi$, $\frac{d\phi}{dw}$, is called the delay).

Application of this criterion leads to Bessel filters, which have excellent transient response but poor frequency selectivity.




The previous criteria stressed behavior at w = 0. The Chebycheff criterion holds that all frequencies in the passband are equally important. Attenuation and attenuation ripple are permitted in the passband, but the maximum attenuation is limited to whatever the designer feels he can tolerate. The Chebycheff filter achieves greater sharpness at the cutoff.




5 Pole Bessel Filter

E. Bessel Filter

The Bessel polynomials in the variable $\frac{1}{s}$ are defined by

$$y_n(\frac{1}{s}) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!(2s)^k} .$$

The polynomials of interest are derived from above by letting

$$h_n(s) = s^n y_n(\frac{1}{s}) = \sum_{k=0}^{n} a_k s^k ,$$

and the transfer function is then given by

$$R(s) = \frac{a_0}{h_n(s)} \; ; \; R(0) = 1 \; .$$

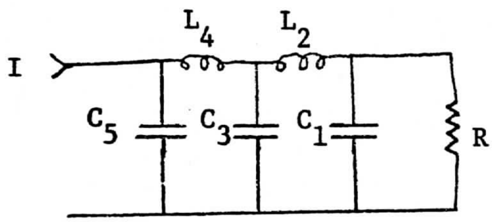
For a 5 pole filter, we have the coefficients

$$a_0 = 945$$
$$a_1 = 945$$
$$a_2 = 420$$
$$a_3 = 105$$
$$a_4 = 15$$
$$a_5 = 1,$$

and the pole locations

$$q_1 = -3.646739$$
$$q_{2,3} = -3.351956 \pm i\ 1.742661$$
$$q_{4,5} = -2.324674 \pm i\ 3.571023,$$

and the configuration

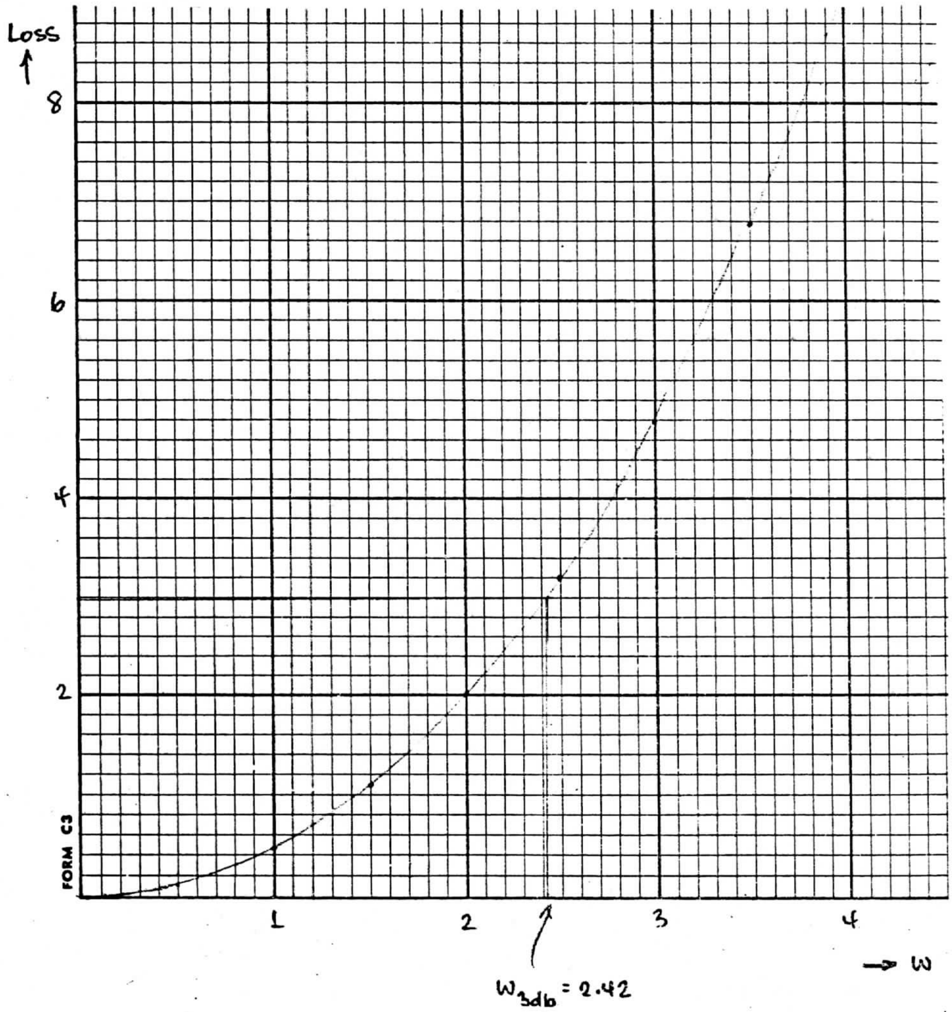



$$C_1 = .0667$$
$$L_2 = .1948$$
$$C_3 = .3103$$
$$L_4 = .4215$$
$$C_5 = .6231$$
$$R = 1 \quad .$$

The loss in dB for the 5 pole Bessel filter can be written

$$\text{Loss} = 20 \log_{10} |R(s)|$$

$$= 20 \log_{10} \left\{ \frac{945}{[945-420w^2+15w^4)^2 + (945w-105w^3+w^5)^2]^{1/2}} \right\} .$$




F. Conversion From Low Pass to Band Pass

A low pass filter is characterized by some distribution of poles and zeros with symmetry about the real axis. If the axis of symmetry is moved to some frequency $w_0$, then $w_0$ will replace zero in the filter response characteristics and a bandpass filter results. Realizing that each pole has a complex conjugate pair, we find that the appropriate transformation is

$$s \rightarrow \frac{s}{w_0} + \frac{w_0}{s} \quad .$$

Since we now have two poles for every previous one, the bandpass filter will have twice as many components as the prototype low-pass filter. This comes about since

$$sL \rightarrow sL + \frac{1}{s(1/w_0^{\,2}L)}$$

which implies that each inductor is transformed into an inductor in series with a capacitor of $1/w_0^{\,2}L$ farads. A similar analysis shows that each capacitor is parallel resonated by an inductance of $1/w_0^{\,2}C$ henrys. Note that the above transformation is not perfect. The pass band is symmetric geometrically, not arithmetically. If $w_1$ and $w_2$ are the edges of the pass band, then the band center is at $w_0 = \sqrt{w_1 w_2}$ and not at $w_0 = (w_1+w_2)/2$.

Thus to convert the 5 pole Bessel low pass filter to a band pass one

(a) determine the desired bandwidth $w_C = w_2 - w_1$ and the desired center frequency $w_0 = \sqrt{w_1 w_2}$

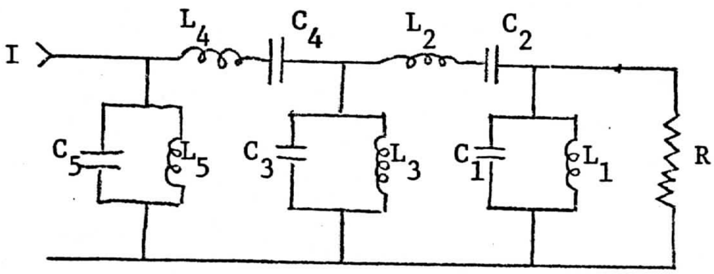
(b) change the bandwidth of the low pass filter to $w_C$

(c) perform the low pass to band pass transformation on the network.

Specifically to convert the 5 pole Bessel 3 db cutoff from 2.42 rad/sec ($w_{max,LP}$) to 26000 Hz and .026 Hz ($f_{max,BP}$ and $f_{min,BP}$) write

$$\frac{w_{LP}}{w_{max,LP}} = \frac{\sqrt{f_{max,BP}f_{min,BP}}}{f_{max,BP}-f_{min,BP}} \left[ \frac{f_{BP}}{\sqrt{f_{max,BP}f_{min,BP}}} - \frac{\sqrt{f_{max,BP}f_{min,BP}}}{f_{BP}} \right]$$

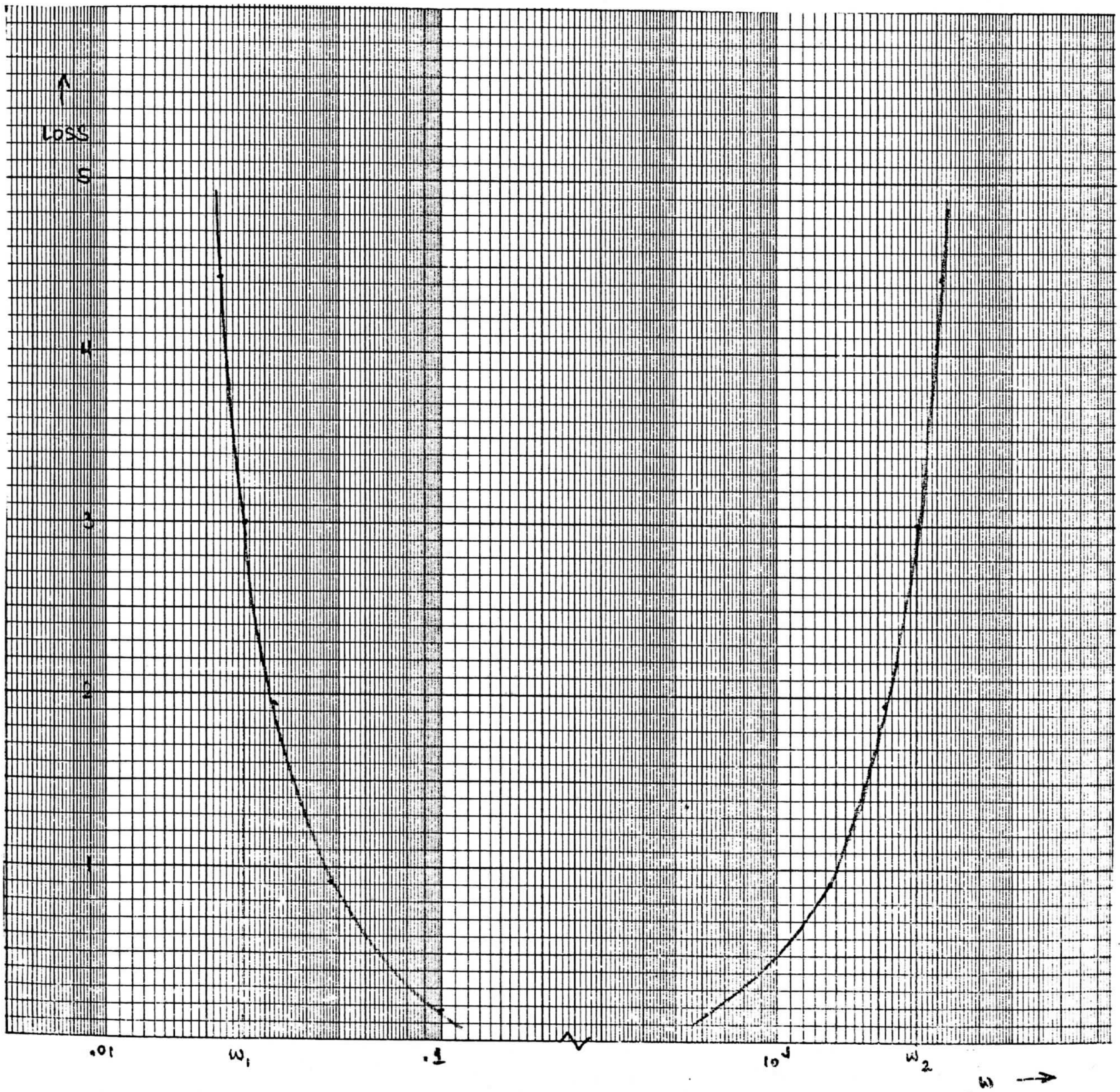
and change the components so that the circuit looks like




$L_5 = 1/w_0^2 C_5$ henry

$C_4 = 1/w_0^2 L_4$ farad

Then the loss curve will have the following form



G. References on Filters

(1) Feynman, Lectures on Physics, Vol. II

(2) Ludkin, Filter Systems and Design

(3) Weinberg, Hughes Aircraft Company Technical Memo #427

(4) Geffe, Simplified Modern Filter Design